Multiple Lenders, Strategic Default and the role of Debt Covenants*

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Abstract

This paper investigates the relationship between competition and contract design in credit markets subject to moral hazard. We consider the stylized representation of the market for loans introduced by Holmstrom and Tirole (1997, 1998), and we explicitly model competition among lenders as an extensive form game. Financial contracts are taken to be non-exclusive, which guarantees that borrowers can trade with several lenders at a time. In such a context, we provide a full characterization of the set of equilibrium allocations and we show that the features of market equilibria crucially depend on the financial contracts made available to financiers. If lenders make use of debt contracts only, the equilibrium outcome is unique, and yields monopoly profits to the aggregate of lenders. If covenants contingent on the project's cash-flow can be included in financial contracts, then every feasible allocations can be supported at equilibrium: market equilibria will be indeterminate and Pareto-ranked. The introduction of institutional mechanisms which prevent borrowers from strategically defaulting on their loans could restore the competitive outcome as the unique equilibrium allocation.

1 Introduction

This paper contributes to the analysis of credit markets where lenders compete to provide funds to borrowers in the presence of moral hazard. We consider the situation where none of the lenders

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can monitor borrowers' transactions with his competitors, which we regard as a salient feature of several modern financial markets. That is, exclusivity of financial relationships cannot be enforced and every single borrower has the opportunity to raise funds from several financiers at a time.

In a moral hazard scenario, non exclusivity of contracts generates a fundamental externality among financiers: The borrowers' effort choices depend on the entire set of loans she takes up, but each of the lenders is only able to monitor his own trades with borrowers. By accepting additional loans, each borrower becomes more willing to misbehave, which reduces the expected repayments to senior creditors, as originally documented by Bizer and DeMarzo (1992). More recently, several works emphasized the perverse effects of such contractual externalities in credit markets. Market equilibria feature lenders earning positive profits and providing a smaller amount of loans as compared with the situation where exclusive clauses can be enforced at no cost (see Parlour and Rajan (2001), Bisin and Guaitoli (2004), Bisin and Rampini (2006) and Bennardo, Pagano, and Piccolo (2009)). The corresponding allocations, however, have typically shown to be constrained efficient: a social planner who does not control either the entrepreneurs' effort choices or their trades *cannot* perform better than markets.[1]

The present paper proposes a further examination of these markets, supporting the view that the set of financial instruments which are available to competitors crucially affects the features of market equilibria. We develop a framework where contracts are non-exclusive but they can be made contingent on publicly observable financial results (cash-flows, assets...). This allows financiers to specify covenants with some targeted investment level, and to design penalties when such covenants are violated. Such instruments are *not* equivalent to controlling the number of traded contracts ex ante, since the ability of investors to punish departures from targeted ratios is reduced by entrepreneurs' limited liability. We fully investigates the strategic role of such covenants on investors' market power and credit rationing for firms. One could in principle argue that an explicit consideration of such additional clauses makes competition more fierce, driving lenders' profits to zero. Quite on the contrary we stress that these additional instruments provide

[1]The efficiency notion corresponding to a situation where the planner cannot enforce exclusivity clauses is referred to as third best optimality. See Bisin and Guaitoli (2004), Attar, Campioni, and Piaser (2006) and Attar and Chassagnon (2009) for a formal definition of this concept and for an extensive discussion of the efficiency results.

new profit opportunities for lenders as well as new threats against their opponents opportunistic behaviors. As a result, market equilibria will be indeterminate and Pareto-ranked.

The possibility that lenders can issue payments' schedule contingent on aggregate variables has typically been disregarded in recent analysis of credit relationships under non exclusivity. In this respect, we show that the (constrained) efficiency features of market equilibria emphasized in these analyses tightly depend on the restriction to the set of financial instruments available to lenders. The presence of additional clauses indeed generates new coordination problems: each of the lenders can use such clauses to protect himself against unilateral deviations by his competitors. This makes possible to support a large set of allocations at equilibrium. Our results therefore suggest that the presence of credit market institutions as well as their regulatory role should be explicitly considered when focusing on non-exclusive credit markets.

The starting point of our analysis is the credit economy considered by Holmstrom and Tirole (1997, 1998), where entrepreneurs need funds to invest in a project which operates under a linear technology subject to moral hazard. With reference to such a standard setting, we model competition amongst financial intermediaries as an extensive form game: lenders simultaneously post their offers; after observing them, borrowers take their portfolio and effort decisions.

We first consider the situation where lenders' offers can be made contingent on the "success" or "failure" of the project, but not on its cash-flow. Since this restriction has been postulated in all recent attempts at modeling non exclusive competition in credit markets, we look at this scenario as our reference benchmark. In such a context, we show that competition delivers an extreme result: The only aggregate allocation supported at equilibrium is the one where lenders earn a monopolistic profit. None of the financiers has a unilateral incentive to propose loans at a smaller rate, because this would always induce the entrepreneur to trade several contracts at a time, and to select the low effort, which makes the deviation not profitable in the first place. One should observe that the possibility to support monopolistic allocations at equilibrium has been emphasized in a number of recent works (see Parlour and Rajan (2001) and Bennardo, Pagano, and Piccolo (2009) among others). Here we show that, as long as exclusivity of contracts cannot be enforced, this extreme non competitive result can also arise in linear environments. In addition, linearity exacerbates the market power of lenders and allows us to pin down monopoly as the unique equilibrium allocation.

This sharp result provides a natural starting point to develop a framework where contracts can be also made contingent on aggregate cash-flows. We interpret these contracts as debt contracts with financial covenants. In practice, covenants are designed to induce efficient decisions from the borrower and thereby reduce potential agency problems between borrowers and lenders. Such financial covenants are typically contingent on verifiable and contractible variables such as balance sheet, income statement or cash flow items[2]. A single lender may gain from the introduction of these covenants because they allow to punish an entrepreneur who deviates from some targeted investment level. Given the entrepreneurs' limited liability, penalties in case of violation of covenants are bounded from above. As a consequence, financiers' ability to write covenants contingent on cash-flows ex-post is not equivalent to controlling for the number of contracts accepted ex-ante. The introduction of these financial covenants, however, does not help lenders to coordinate on some competitive outcome: We prove that every feasible allocation can be sustained as a pure strategy equilibrium of our competition game.

This calls for the explicit consideration of some institutional mechanism dealing with such a coordination issue. In particular, we discuss to what extent financial institutions can induce lenders to coordinate on a zero-profit, constrained efficient allocation. The specific contractual externality arising in our setting creates incentives for borrowers to make "false" promises, i.e. to accept more loans than those that can effectively be repaid. We consider the role performed by a court that can verify payments to the lenders and has the ability to punish the entrepreneur if it appears that she made false promises. While the presence of such an institution does not affect the equilibrium outcome when only debt contracts are allowed, it sharply modifies the nature of competition when contracts can include covenants on investment. In that case, we argue that the only equilibrium allocation is the competitive one. Our analysis thus provides a rationale for the joint use of covenants in debt contracts and institutions that monitor the level of indebtedness of firms.

[2]See for example Paglia and Mullineaux (2006), Demiroglu and James (2008).

2 The model

To develop our intuitions, we build on the stylized representation of the credit market popularized by Holmstrom and Tirole (1997, 1998), and Tirole (2006) .

Agents, technology and preferences. We consider a production economy that lasts two periods. It is populated by a single representative entrepreneur and a finite number N of investors. The entrepreneur owns a project and can ask money to lenders to expand the scale of her project. Production takes place through a linear technology which realizations are subject to uncertainty over two aggregate states denoted "success" and "failure". An investment of $I \in \mathbb{R}_+$ yields a final output (cash-flow) of GI, with $G \in \mathbb{R}_+$, in case of success and of zero in case of failure. Output can be verified at no cost. The probability distribution over final outcomes depends on an unobservable effort $e = \{L, H\}$ chosen by the entrepreneur. Denote $(\pi_e, 1 - \pi_e)$ the probability distribution induced by the effort choice e, where π_e is taken to be the probability of success. We let $e = H$ represent the high level of effort, which guarantees that $\pi_H > \pi_L$. If the entrepreneur misbehaves selecting $e = L$, she receives a private benefit $B \in \mathbb{R}_+$ per unit invested in the project. In line with Holmstrom and Tirole (1998), we introduce the following two assumptions. First, the investment project has a positive net present value if and only if the entrepreneur selects $e = H$; that is:

$$\pi_H G > 1 \text{ and } \pi_L G + B < 1, \tag{1}$$

Second, if the entrepreneur chooses $e = H$, the unitary revenue from the project is smaller than the unitary agency cost; that is:

$$0 < \pi_H G - \frac{\pi_H B}{\Delta\pi} < 1, \tag{2}$$

where $\Delta\pi = \pi_H - \pi_L$.

The entrepreneur is risk-neutral and protected by limited liability. She has an initial endowment of $A \in \mathbb{R}_+$ and can raise additional funds by trading financial contracts issued by competing investors. If she gets a credit of I units in exchange for an aggregate repayment of R in the case of success,[3] her expected utility is:

$$U(I,R,e) = \begin{cases} \pi_H \max\left\{\big(G(I+A)-R\big),0\right\} - A & \text{if } e = H \\ \pi_L \max\left\{\big(G(I+A)-R\big),0\right\} + B(I+A) - A & \text{if } e = L \end{cases}$$

If $G(I+A) < R$, then default takes place and the entrepreneur can get a positive payoff only selecting the low effort $e = L$ and earning the private benefit $B(I+A)$.[4] If the entrepreneur decides not to enter into a credit relationship, she is left with the option to carry out the production process using her endowment only. The corresponding reservation payoff is $U(0) = (\pi_H G - 1)A$, which is strictly positive given (1).

For every aggregate allocation (I,R) traded by the entrepreneur, we let $e(I,R) \in \arg\max_e U(I,R,e)$ be any corresponding optimal effort choice. We denote $\mathcal{H} = \{(I,R) \in \mathbb{R}^2_+ \,:\, e(I,R) = H\}$ the set of aggregate allocations inducing $e = H$ as an optimal effort choice, and $\mathcal{L} = \{(I,R) \in \mathbb{R}^2_+ \,:\, e(I,R)\}$ its complement. Finally, $\Psi = \mathcal{H} \cap \mathcal{L} = \big\{(I,R) \in \mathbb{R}^2_+ \,:\, \pi_H\big(G(I+A)-R\big) = \pi_L\big(G(I+A)-R\big) + B(I+A)\big\}$ is the set of aggregate allocations that make the entrepreneur indifferent between $e = H$ and $e = L$. We call Ψ the incentive frontier.

The payoff to lender i only depends on his trades (I^i, R^i) with the entrepreneur. Since lenders are risk-neutral, one gets:

$$V^i(I^i, R^i, e) = \pi_e R^i - I^i \quad \text{with } \; e \in \{L, H\},$$

and his reservation utility is set to zero.

If default takes place, a single lender may not be repaid according to the contractual premises, and his payoff will be determined by the relevant seniority rule, as we shall explain below.

[3] Standard arguments guarantee that aggregate payments must always be set equal to zero in the case of failure.

[4] One should also observe that A has been taken to be observable, so that if the entrepreneur chooses $e = L$, he has to invest A. Assuming that the entrepreneur does not invest A if he exerts effort $e = L$ does not modify the results.

Contracts and strategic default. Lenders compete by offering non-exclusive, bilateral contracts to the single borrower. We take a financial contract proposed by lender i to be the array $(I^i, R^i(\theta))$, where $I^i \in \mathbb{R}_+$ is a credit amount and $R^i(\theta) : \mathbb{R}_+ \to \mathbb{R}_+$ is a schedule of repayments contingent on the cash-flow realization $\theta = \{0, GI\}$ $\forall I \in \mathbb{R}_+$. Given the entrepreneur's limited liability, one necessarily has $R^i(0) = 0$; that is, each lender i will not be asking for any repayment if the project fails. To the extent that the final cash-flow perfectly reflects the total investment I in case of success, this amounts to contracting on I: Without loss of generality, we hence say that the schedule $R^i(I)$ associates any possible aggregate loan taken up by the entrepreneur to a repayment for lender i. [5]

The entrepreneur can simultaneously trade with several lenders, optimally choosing the amount of loans and the repayments to be made given the array of offers. In our setting, though, $R^i(I)$ will not necessarily coincide with the payment received by lender i if the borrower takes an aggregate loan of I and the project succeeds, because the entrepreneur can strategically default. This happens whenever she commits to repay more than the cash-flow generated by the project in case of success. In this case, the entrepreneur makes 'false promises', entering into multiple credit relationships even though not all repayments will be honored. The possibility of strategic default is a natural implication of non-exclusive competition: When the entrepreneur trades with multiple lenders, she might have an incentive to accept contracts involving conflicting prescriptions. Financial contracts could in principle specify seniority rules to determine the order of repayments in case of default. However, such clauses do not always preclude conflicts. For instance, it can happen that two accepted contracts entail the same level of seniority.[6] For this reason, we do not treat priority rules as given,[7] but we determine them as part of the entrepreneur's optimal behavior. The following timing of events describes our non-exclusive competition game:

[5]This is not equivalent to contracting on the number of contracts accepted ex ante, because of the entrepreneur's limited liability. This limits the set of available punishments if the entrepreneur deviates from the targeted investment level.

[6]As anecdotal evidence, consider the example of ABN Amro and JPMorgan who disputed over their claim on Dutch investor Louis Reijtenbagh's art collection after he defaulted on both institutions' loans. Reijtenbagh apparently used his art collection as collateral in the two banks.

[7]This contrasts with Bennardo, Pagano, and Piccolo (2009), who focus mainly on pro-rata rules, and with Bizer and DeMarzo (1992) who impose priority rules.

1. Each lender i proposes a financial contract, i.e. an array $(I^i, R^i(.))$.

2. Having observed the array of offered contracts, the entrepreneur chooses a subset of lenders to trade with and she takes the corresponding optimal effort. In addition, if her portfolio decisions induce a strategic default, she determines a priority rule for lenders.

3. Cash flow are realized and payments are made.

Throughout the paper, and unless stated otherwise, the equilibrium concept is pure strategy subgame perfect equilibrium.

Observe that strategic default will never take place at equilibrium. In our setting, an incentive to strategically defaulting arises since the entrepreneur always has the opportunity to earn a positive private benefit on the whole array of offered contracts. Whenever defaulting, the entrepreneur will hence be choosing $e = L$. However, (1) guarantees that if $e = L$ is selected then at least one lender will earn a strictly negative payoff. At any pure strategy equilibrium, one must hence have $e = H$.

Constrained efficiency: competitive and monopoly allocations. We introduce a pair of aggregate allocations that will be useful benchmarks for our analysis. The competitive allocation (I^c, R^c) maximizes the entrepreneur's utility subject to incentive and participation constraints. Given (1) and (2), (I^c, R^c) solves:

$$(I^c, R^c) \in \arg\max_{(I,R)} U(I, R, e)$$

$$\text{s.t.} \qquad U(I, R, H) \geq U(I, R, L) \tag{3}$$

$$\pi_H R - I \geq 0, \tag{4}$$

where (3) is the entrepreneur's incentive compatibility constraint, and (4) the (aggregate) participation constraint of lenders. At the optimum, (3) and (4) are binding. The optimal investment-repayment pair is therefore

$$I^c = A\Big(\frac{G - \frac{B}{\Delta\pi}}{\frac{B}{\Delta\pi} - G + \frac{1}{\pi_H}}\Big) \qquad \text{and} \qquad R^c = \frac{1}{\pi_H} I^c \tag{5}$$

Similarly, define the monopolistic allocation (I^m, R^m) as that prevailing when financial investors maximize their joint utility. It is determined by

$$(I^m, R^m) \in \arg\max_{(I,R)} \; \pi_H R - I$$
$$\text{s.t.} \qquad U(I,R,H) \geq U(I,R,L)$$
$$U(I,R,H) \geq U(0)$$

which implies that

$$I^m = A\Big(\frac{G\Delta\pi}{B} - 1\Big) \qquad \text{and} \qquad R^m = GI^m. \tag{6}$$

Both the competitive and the monopoly allocations are feasible choices for a planner who cannot observe borrower's effort and retains a full control of her trades. The whole set of constrained efficient (I, R) allocations is identified by the linear relationship

$$R = \big(G - \frac{B}{\Delta\pi}\big)(I + A) \tag{7}$$

3 Credit market equilibrium with plain debt contracts

This section analyzes the situation where lenders propose contracts which can be made contingent on the aggregate state success or failure state, but *not* on the total cash-flow available in the success state. We denote these contracts plain debt contracts. A plain debt contract offered by lender i is any array $C_i = (I_i, R_i) \in \mathbb{R}^2_+$, where I_i is a loan amount, and R_i is the corresponding fixed repayment. It is useful for the analysis to denote $p_i = \dfrac{R_i}{I_i}$ the price of this contract.

We provide here a full characterization of the set of pure strategy equilibria of the competition game where lenders are restricted to make use of plain debt contracts only. Observe that such a restriction has typically been postulated in the literature on non-exclusive credit markets subject to moral hazard: Bizer and DeMarzo (1992), Parlour and Rajan (2001), Bisin and Guaitoli (2004), Bisin and Rampini (2006), Attar, Campioni, and Piaser (2006), and Bennardo, Pagano, and Piccolo (2009). By introducing our reference benchmark, the present section is therefore also providing a direct comparison with these researches.

Our first step is to emphasize a simple property of the entrepreneur's optimal choices. Take any array of lenders' offers and let $C = (I, R)$ be the corresponding aggregate investment-repayment pair optimally chosen by the entrepreneur. Also, denote $\tau(C)$ the entrepreneur's marginal rate of substitution evaluated at C. For a given level of effort, $\tau(C)$ reflects the maximum price that the entrepreneur is willing to pay for an additional unit of investment for his utility to remain constant. In particular, one gets $\tau(C) = G$ for every $C \in \mathcal{H}$ and $\tau(C) = G + \frac{B}{\pi_L}$ for every $C \in \mathcal{L}$. In words, the entrepreneur does not modify her payoff by trading additional contracts of price $p = G$ ($p = G + \frac{B}{\pi_L}$) and selecting the effort $e = H$ ($e = L$).
Given the linearity in preferences, the entrepreneur always has an incentive to trade contracts which price is strictly smaller than her marginal rate of substitution. This is formalized in the following:

Lemma 1 *Let $\{C_1 = (I_1, R_1), \cdots, C_n = (I_n, R_n)\}$ be any array of offers such that the corresponding investment-repayment pair $C = (I, R)$ optimally chosen by the entrepreneur does not lie on the incentive frontier Ψ. Then, if default does not take place, each contract C_i such that $p_i < \tau(C)$ is accepted, and each contract C_i such that $p_i > \tau(C)$ is rejected. If default occurs, each offer $C_i = (I_i, R_i)$ such that $I_i > 0$ is accepted by the entrepreneur.*

As long as default does not take place, the entrepreneur will accept (reject) all contracts issued at a price strictly smaller (greater) than her corresponding marginal rate of substitution. If her behavior induces default, though, the entrepreneur will accept all contracts involving a strictly positive level of credit, since this increases her private benefit.
Lemma 1 has one main implication. Since the marginal rate of substitution evaluated in $\mathcal{L}$ is greater than that evaluated in $\mathcal{H}$, each contract accepted if $e = H$ is chosen is also accepted if

$e = L$ is chosen. More precisely: at any optimal choice, all contracts of price strictly smaller than G will necessarily be accepted. This intuition is crucially exploited to argue that in our competition game there is always a pure strategy equilibrium. In particular, the following proposition shows that the monopolistic allocation can be supported at equilibrium.

Proposition 1 *The competing lenders game has always an equilibrium where one of the lenders proposes* (I^m, R^m)*, and all the others propose the null contract* $(0, 0)$*. At equilibrium, the entrepreneur accepts the offer* (I^m, R^m) *and selects* $e = H$*.*

Proposition 1 shows that there always exists an equilibrium with only one active lender earning the monopoly profit. The intuition for the result is as follows. Following any deviation by one of the inactive lenders, the entrepreneur always has an incentive to accept the deviating contract together with the monopolistic one proposed by the incumbent and to exert low effort. In such a case, given conditions (1) and (2), it is impossible for any deviating lender to gain as long as all contracts are repaid according to their promises. A deviation may hence be profitable only by inducing the entrepreneur to default over the aggregate of loans. However, one can always construct the entrepreneur strategy in terms of her priority rules in such a way to make all these deviations non profitable. In particular, we show that both the situation first repays the (monopolistic) incumbent lender and that where lenders are repaid according to some pro-rata rules are consistent with the equilibrium.

The result of proposition 1 stands in sharp contrast with that arising in a standard setting where exclusive clauses are enforceable at no cost. In this case, each inactive lender can profitably deviate proposing a contract which provide the entrepreneur with a payoff strictly greater than the equilibrium one and inducing her to select the high level of effort. It follows that the unique allocation supported at equilibrium is the competitive allocation (I^c, R^c), where the borrower appropriates the whole surplus.

The possibility to support monopolistic equilibria in credit markets subject to moral hazard has already been documented in several recent works on non-exclusive credit markets subject to moral hazard (see Parlour and Rajan (2001) and Bennardo, Pagano, and Piccolo (2009)) with concave production function and in presence of a sufficient number of investors. Our first contribution is hence to show that this result can also arise in a setting with a linear production technology and irrespectively of the number of investors.

In the next paragraphs we show that the monopoly is the *only* aggregate allocation that can be supported at a pure strategy equilibrium of our competition game. The result is a direct implication of the next propositions 2 and 3, which identify some necessary conditions on the set of contracts offered at any pure strategy equilibrium of our competitive game. This is the second contribution of the present section.

Observe first that given aggregate allocation $C \in \mathcal{H} - \Psi$, where the borrower strictly prefers the effort choice $e = H$, there are incentives for additional trades. That is, (1) and (2) guarantee that if the additional contract $\big(x, (1/\pi_H + \epsilon)x\big)$ were introduced, where x and ϵ are sufficiently small, the borrower will strictly prefer trading this contract together with the C allocation rather than trading C alone. Furthermore, as long as $e = H$ is selected, the corresponding increase in the social surplus will be strictly greater than that in the borrower's payoff. Every single lender i could therefore profitably deviate offering $\big(I_i + x, R_i + (1/\pi_H + \epsilon)x\big)$ instead of (I_i, R_i). For the allocation C to be supported at equilibrium, it must hence be that, following such a deviation, the borrower has a strict incentive to select $e = L$. Since both x and ϵ can be taken to be arbitrarily small, one can directly derive some implications on the features of contracts offered at equilibrium. This is done in the following

Proposition 2 *If $C = (I, R) \in \mathcal{H}$ is an equilibrium allocation, then the array of contracts offered at equilibrium must be such that:*

(i) At least one aggregate allocation $\tilde{C} = (\tilde{I}, \tilde{R}) \in \mathcal{L}$ such that $U(C, e = H) = U(\tilde{C}, e = L)$ must be available to the borrower.

(ii) In particular, the allocation $\bar{C} \in \Psi$ such that $U(C, e = H) = U(\bar{C}, e = H)$ must be always available to the borrower.

The intuition for the result is as follows. If no $\tilde{C}$ allocation were available, then it would always be possible for a single lender to profitably deviate offering some (small) additional loan without triggering the low effort choice. It follows that each allocation $C \in \mathcal{H} - \Psi$ can be supported at equilibrium only having some inactive lenders who guarantee the availability of $\tilde{C}$. At the equilibrium, each of those inactive lenders must have no unilateral incentive to deviate. That is, at least one allocation $\tilde{C}$ must remain available if a single lender withdraws his offer. The only aggregate allocation that satisfies such a requirement is the allocation $\bar{C}$ defined by (ii). [8]

Proposition 2 provides several insights on the structure of contracts that will be offered at any equilibrium of our competitive game. In particular, it suggests that no contract C_i of price $p_i \in (G, G + \frac{B}{\pi_L})$ will be offered, otherwise the borrower would trade them together with $\bar{C}$ and select $e = L$. A main implication of proposition 2 is then given by the following:

Corollary 1 *If $C = (I, R)$ is an equilibrium allocation, then $C \in \Psi$.*

That is, every equilibrium allocation belongs to the incentive frontier. The proof of corollary 1 stresses the fact that, if some $C \in H - \Psi$ were supported a equilibrium, then, given the information available on the set of equilibrium offers, there would always be incentives for unilateral deviations inducing the borrower to select $e = H$.

In a next step, we consider deviations where a lender who is active at the equilibrium tries to gain by raising the price of his contract. The following proposition argues that a necessary condition for these deviations not to be profitable is that all contracts which are traded at equilibrium have a price equal to the corresponding marginal rate of substitution of G. That is:

Proposition 3 *If the contract $C_i = (I_i, R_i)$ is accepted at an equilibrium, then necessarily $p_i = \dfrac{R_i}{I_i} = G$.*

The intuition for the result is as follows. Suppose that a contract C_i of price $p_i < G$ is traded at equilibrium. Then, it is always possible for lender i to deviate towards an alternative contract of

[8] If $C \in \Psi$, then the proposition is verified letting $C = \tilde{C} = \bar{C}$.

price strictly greater than p_i without triggering low effort. Such a deviation involves a reduction in the amount of loans and in the associated social surplus, and it is profitable as long as the borrower decides to accept it. To block the deviation, the equilibrium utility must necessarily remain available to the borrower if any of the lenders withdraws his offer. In our linear scenario, this amounts at requiring that all traded contracts must have a price equal to the corresponding marginal rate of substitution G. It then follows from lemma 1 that no contract of price $p_i < G$ will be offered.

Every pure strategy equilibrium of our competition game will therefore have the entrepreneur earning her reservation utility, and the aggregate of lenders getting a positive profit. In addition, all contracts will be offered at price equal to the marginal rate of substitution, i.e. $p_i = \{G, G + \frac{B}{\pi_L}\}$, $\forall i \in N$.[9]

Now, it is a direct implication of corollary 1 and proposition 3 that only the monopolistic allocation (I^m, R^m) will be supported at equilibrium.

Corollary 2 *The monopoly allocation (I^m, R^m) is the unique equilibrium allocation of the competition game.*

The monopoly allocation (I^m, R^m) is the only allocation on the incentive frontier Ψ that can be attained when the price of every traded contract is equal to G.

4 Credit market equilibrium with covenants contingent on cash-flows

The essential feature of markets where lenders compete through non-exclusive contracts is that none of the competitors can make his offers contingent on the proposals of his rivals. In the context of credit relationships, this has often translated into assuming that financial contracts cannot be contingent on total investment, or total assets (see for instance Bizer and DeMarzo (1992), Bisin and Guaitoli (2004) or Bisin and Rampini (2006)). This is the approach we followed in section 3.

[9]Observe that there could always be some C_i contract offered at a price $p_i > G + \frac{B}{\pi_L}$, but, following lemma 1, these contracts will never be traded unless default takes place.

We now extend our analysis by assuming that financial contracts can be contingent on the entrepreneur's total cash-flow which is itself related to the total amount invested initially. There are indeed several economic situations where agents' trades cannot be monitored and the corresponding aggregate outcomes can be observed. In a production context, firms' cash-flows or assets are verifiable and contractible.

The main implication of expanding the set of contracts available to each of the lenders is to provide them with an additional set of punishments, or coercive clauses, in case the entrepreneur deviates from some targeted investment level. It is thus natural to interpret such contingent contracts as debt contracts with covenants. The question is whether one's ability to write covenants contingent on cash-flows ex post is equivalent to one's ability to control for the number of contracts accepted ex ante, as it would be the case in a setting of exclusive competition. Intuitively, the two might differ because investors' ability to "punish" departures from targeted investment levels is limited. In our model, the entrepreneur's limited liability sets an upper bound on penalties that lenders can impose.

Considering contracts contingent on total cash-flow amounts to conditioning repayment R_i on total investment I rather than on the single investment I_i proposed by investor i. In contrast, exclusive competition amounts to conditioning I_i on total investment I before investment is actual sunk. The next proposition states that introducing cash-flow-contingent covenants dramatically changes the set of equilibrium allocations in our competition game.

Proposition 4 *Take any aggregate allocation $C = (I, R) \in \mathcal{H}$ that is feasible, i.e. $R \in [\frac{1}{\pi_H} I, GI]$. If repayments R_i can be contingent on the total investment $I = \sum_{i \in \mathcal{I}} I_i$ chosen by the entrepreneur, any such allocation can be supported at equilibrium as long as the number of investors N is large enough and the private benefit of shirking B is large enough, in the sense that $G < \frac{1}{\pi_H}(1 + B)$.*

Proposition 4 establishes that each feasible allocation can be sustained at equilibrium with covenants contingent on the final cash-flow. This result holds under some assumptions regarding the number of agents in the economy, and the severity of the moral hazard problem. The corresponding equilibrium strategies can be described as follows. A first set of lenders offer contracts that collectively grant the entrepreneur his equilibrium utility: These contracts are active, i.e. accepted at equilibrium. To achieve this, each offer is formulated so that the entrepreneur obtains his equilibrium utility when accepting all (or all but one) contracts: If the entrepreneur accepts any other set of contracts, she is punished by having to repay all the cash-flow. Such contracts can be interpreted as debt contracts, with covenants specifying a targeted investment level. When a covenant is violated, lenders capture the firm's assets.[10] An important feature is that any equilibrium utility can be sustained by this set of active contracts. A second set of lenders offer high price contracts that are not accepted at equilibrium. Such passive contracts ensure that the entrepreneur would obtain his equilibrium utility, were she to accept all contracts and exert effort $e = L$.

If agents select such strategies, every feasible allocation can be supported at equilibrium. Firstly, the entrepreneur always has an incentive to only accept the offers of the first group of lenders and to choose $e = H$. Secondly, no lender has a unilateral incentive to deviate, irrespectively of his equilibrium profits. The intuition goes as follows. A single lender cannot gain by offering a contract if this is the only one traded by the entrepreneur at the deviation stage: in this case, the amount of loans will necessarily be large, which provides incentives to the entrepreneur to default on all offered contracts. Given the covenants issued at equilibrium, a single lender cannot deviate by offering a contract if it is traded together with additional offers.

In our construction, market equilibria are supported by a large number of inactive lenders. This is because the credit individually proposed by each inactive lender needs to be small, while the amount of loans available in the aggregate needs to be large. A second requirement is that B has to be large. This is because all equilibria rely on the threat that any deviation triggers strategic default and therefore low effort: this threat has to be credible, i.e. the private benefit of shirking has to be larger than the per unit monetary return of the project. Finally, the equilibria are constructed so that a deviating investor cannot make profits if the entrepreneur chooses $e = L$. This is done by assuming, as in section 3 that the deviating investor is repaid after the others, in case of strategic default. The result also holds when investors are repaid according to a pro rata rule. With no

[10] In our model, asset value is simply the final realized cash-flow.

doubt, priority rules matter for equilibria to exist. If one investor could deviate and make his contract senior to any other, such deviations would be hard to deter.

The above discussion sheds light on the strategic role of covenants. When the contract space is reduced, as it is the case when only the state of nature can be contracted upon, a unique equilibrium can be sustained that achieves the monopoly allocation. By contrast, expanding the contract space creates an indeterminacy that can be explained as follows. On the one hand, introducing cash-flow contingent covenants increases the ability of investors to punish deviations, which should enhance competition. On the other hand, cash-flow contingent contracts make coordination easier, and deter entry of passive investors, which renders all feasible allocations sustainable.

The indeterminacy stated in proposition 4 has important consequences for credit market efficiency. While section 3 determines a unique efficient equilibrium corresponding to the monopoly allocation, inefficient equilibria can be sustained here (i.e. equilibria such that the allocation $C = (I, R)$ is not on Ψ). This suggests the need to consider some institutional mechanism coordianting lenders' behaviors.

5 Institutional constraints to support the competitive outcome

A main feature of our credit economy is the lack of any institution which prevents the entrepreneur from strategically defaulting over the loans she accepts. That is, the entrepreneur has the opportunity to make "false " promises, accepting a number of contracts greater than those he will be able to honor. In such cases, even before production is realized, the entrepreneur knows that she will be bankrupt with certainty. The ability to default strategically influences the nature of competition, because some lenders might have an interest to induce a default, and benefit from it if their claim is senior to that of others.

In order to clarify the strategic role of this decision, consider a situation where strategic default is ruled out. That is, the array of contracts accepted by a single borrower must satisfy the inequality$\sum_{i\in\mathcal{I}}(GI_i - R_i) + GA \geq 0$. That is, the entrepreneur cannot accept an array of contracts such that the sum of repayments is greater than the maximum achievable cash-flow. Such an assumption modifies the nature of competition among financiers, as stated in the following proposition.

Proposition 5 *The following holds.*

(i) If repayments R_i are contingent on the total investment $I = \sum_{i \in \mathcal{I}} I_i$ chosen by the entrepreneur, and if strategic default is precluded in the sense that accepted contracts are such that $\sum_{i \in \mathcal{I}}(GI_i - R_i) + GA \geq 0$, then the competitive allocation $C^c = (I^c, R^c)$ is the unique equilibrium allocation.

(ii) If only plain debt contracts are allowed, then the monopoly allocation (I^m, R^m) is the unique equilibrium allocation.

Ruling out strategic default when contracts are contingent on the final cash-flow removes the equilibrium indeterminacy and leads to the competitive allocation as the unique equilibrium allocation. It is worth pointing out that when only plain debt contracts are allowed, precluding strategic default does not affect our previous result: the monopoly allocation still remains the unique equilibrium allocation. The intuition goes as follows. Introducing contingent covenants and precluding strategic default allows to restore the basic mechanism of price competition. A cash-flow contingent contract allows an investor to deviate towards a contract that is profitable as long as it is the only one accepted by the entrepreneur. If strategic default is precluded, there is always room for a single deviator to serve the whole market by marginally undercutting the equilibrium offers without triggering $e = L$. This indeed supports the competitive outcome at equilibrium. The same situation cannot arise if strategic default is allowed because following each unilateral deviations the entrepreneur's incentive to accept all contracts and shirk.

A natural idea to preclude strategic default is to impose unlimited liability to the entrepreneur if she makes false promises. In our setting this can be achieved through a court entered by a lender after he has received his payments. The court can verify at some positive cost all payments received by lenders and has the ability to effectively punish the entrepreneur if contractual promises have not been honored.[11] At equilibrium, the court effectively acts as a threat that counters the temptation of the entrepreneur to undertake a strategic default. Financial institutions in the spirit of clearinghouses can also play a similar role. Consider for instance the situation where, after contracts have been accepted but before production has realized, a clearinghouse requires the entrepreneur a good faith deposit. In a next step, the clearinghouse observes the whole array of payments and pays back the deposit to the entrepreneur only if strategic default has not occurred. Such a financial institution should reduce the entrepreneur's incentives to make, being thereby instrumental to overcome equilibrium indeterminacy and to support constrained efficient allocations at equilibrium.

6 Conclusion

In this paper, we have presented a model of a credit market in which lenders compete to provide funds to a single borrower who is taking some unobservable actions. We consider a situation where competition is non-exclusive: lenders cannot monitor each other trades with the borrower. In such a scenario, we argue that the feature of market equilibria crucially depend on the set of financial instruments made available to lenders.

If lenders were restricted lenders to make use of simple debt contracts, i.e. contracts only contingent on the project's success or failure, then the only aggregate allocation supported at equilibrium yields a monopolistic profit to financiers. However, if lenders can condition their required payments on the project's cash-flows, then a new set of strategic interactions is introduced. As a result, market equilibria will be indeterminate and Pareto-ranked.

The fact that market outcomes crucially depend on the instruments available to financiers suggests that the positive and normative implications of non-exclusivity in credit markets should be carefully examined. Recent researches in this area have stressed that market equilibria can involve positive profits for the active financiers. At the same time, they have stated a constrained efficiency result: A social planner who does not control either entrepreneur's effort choices or

[11]The court can for instance send the borrower to prison ruining her reputation. The court can also punish the lender if his complaint against the entrepreneur is not justified.

her trades *does not* indeed perform better than markets (see Bizer and DeMarzo (1992), Parlour and Rajan (2001), Bisin and Guaitoli (2004), Bisin and Rampini (2006)). Our analysis indicates that this last conclusion importantly depends on the restriction to debt contracts which has been postulated in all these contributions.

We also suggest that credit market's institutions play a fundamental role to sustain coordination among lenders. In particular, designing institutions that prevent borrowers' from strategically defaulting could restore a fully competitive outcome.

Appendix

Proof of Lemma 1

The proof is developed by contradiction. Let $C_i = (I_i, R_i)$ be a contract not traded by the entrepreneur at her optimal choice C. If $p_i < \tau(C)$, then one can directly check that

$$U\big(I + I_i, R + R_i, e(I, R)\big) > U\big(I, R, e(I, R)\big).$$

That is, the entrepreneur always has an incentive to trade the C_i contract, without changing her effort choice. This contradicts that C is an optimal choice for the entrepreneur. A similar argument can be used to show that all offers C_i with $p_i > \tau(C)$ are not traded by the entrepreneur.

■

Proof of Proposition 1.

Let $(I_1, R_1) = (I^m, R^m)$ and $(I_i, R_i) = (0, 0)$ for $i = 2, \ldots, n$. Then, it is a best reply for the entrepreneur to trade the contract (I_1, R_1) and to select $e = H$. This in turn provides her with the reservation utility $U(I_1, R_1, e = H) = \pi_H GA - A$. We now show that none of the lenders has a unilateral incentive to deviate. Since lender 1 is earning the monopoly profit, only deviations of the inactive lenders must be considered. Let (I'_2, R'_2) be a deviation of any of them, say lender 2.

Suppose first that this deviation induces the entrepreneur to select $e = H$. In this case, for the contract (I'_2, R'_2) to yield a strictly positive profit it must be $R'_2 > \frac{1}{\pi_H} I'_2$. We argue that following any such deviation the entrepreneur has indeed an incentive to select $e = L$. Let K be the aggregate allocation optimally selected by the entrepreneur at the deviation stage, and assume that $K \in \mathcal{H}$. We first show that, following any unilateral deviation of lender 2, one has $K = (I'_2, R'_2)$; that is, the only contract traded by the entrepreneur at the deviation stage will be (I'_2, R'_2). Assume, on the contrary, that the entrepreneur accepts both (I_1, R_1) and (I'_2, R'_2). Since (I_1, R_1) belongs to the set Ψ, we have:

$$G(I_1 + I'_2 + A) - (R_1 + R'_2) = \frac{B(I_1 + A)}{\Delta\pi} + \big(G\, I'_2 - R'_2\big).$$

By (2), $\dfrac{R'_2}{I'_2} > \dfrac{1}{\pi_H} > G - \dfrac{B}{\Delta\pi}$. It follows that:

$$G(I_1 + I'_2 + A) - (R_1 + R'_2) < \frac{B(I_1 + A)}{\Delta\pi} + G\, I_2 - (G - \frac{B}{\Delta\pi})\, I_2 = \frac{B(I_1 + I_2 + A)}{\Delta\pi},$$

i.e. $(I_1 + I'_2, R_1 + R'_2) \in \mathcal{L}$. One hence gets: $K = (I'_2, R'_2) \in \mathcal{H}$.

By continuity, it is then always possible to find a $\mu \in [0, 1[$ such that:

$$\pi_H\big(G(\mu I_1 + I'_2 + A) - (\mu R_1 + R'_2)\big) = \pi_L\big(G(\mu I_1 + I'_2 + A) - (\mu R_1 + R'_2)\big) + B(\mu I_1 + I'_2 + A). \quad (8)$$

That is, if the entrepreneur took a loan of $(\mu I_1 + I'_2)$ paying back the amount $(\mu R_1 + R'_2)$ to the aggregate of investors, she would be indifferent between selecting $e = L$ and $e = H$ (recall that $\mu < 1$ because $(I_1 + I'_2, R_1 + R'_2) \in \mathcal{L}$). Since $p_1 = \dfrac{R_1}{I_1} < G + \dfrac{B}{\pi_L}$, one gets:

$$U(I_1 + I'_2, R_1 + R'_2, e = L) > U(\mu I_1 + I'_2, \mu R_1 + R'_2, e = L) = U(I'_2, R'_2, e = H).$$

It follows that $K = (I'_2, R'_2) \in \mathcal{H}$ will not be an optimal choice: the entrepreneur strictly prefers $e = L$, which contradicts our assumption.

Suppose next that, following the deviation (I'_2, R'_2), the entrepreneur chooses $e = L$. Given (2), the deviation can hence be profitable only if the entrepreneur decides to strategically default[12] in which case she will accept all offered contracts. Given (I'_2, R'_2), the entrepreneur will hence select $e = L$ whenever the payoff she gets by defaulting is greater than her equilibrium utility

$$B(I^m + A + I'_2) \geq \pi_H(G(I^m + A) - R^m) = \frac{\pi_H B}{\Delta\pi}(I^m + A). \quad (9)$$

Given the definition of I^m, the inequality can be rewritten as

$$I'_2 \geq \frac{\pi_L}{\Delta\pi}(I^m + A) = \frac{\pi_L G A}{B} \quad (10)$$

[12]It follows from (2) that $\pi_L G - 1 < 0$, which implies that any lender who is repaid according to the contractual premises earns a strictly negative profit if $e = L$ is selected.

Suppose that I_2' satisfies (10), then the entrepreneur's strategy can always be constructed in such a way that the deviation is not profitable. Fix the following priority rule for the entrepreneur: lender 1 is repaid first, and lender 2 is the residual claimant. Lender 2 will then earn a strictly positive profit whenever

$$\pi_L \left(G(I^m + I_2' + A) - R^m\right) - I_2' > 0 \quad \Longleftrightarrow \quad I_2' < \frac{\pi_L G A}{1 - \pi_L G}. \tag{11}$$

It can be directly checked that, given (1), the inequalities (10) and (11) will never hold simultaneously, which completes the proof of Proposition 1.

Finally, one should observe that there are several entrepreneur's behaviors in terms of her liquidation decisions being consistent with the equilibrium. Take as an example the situation where the entrepreneur decides to repay all defaulted loans are repaid according to some *pro-rata* rule. In this case, (11) becomes $\pi_L G(I^m + I_2' + A)\dfrac{I_2'}{I^m + I_2'} > I_2'$. A direct computation shows that this latter inequality is not compatible with condition (10).[13]

■

Proof of Proposition 2

Proof of assertion (i).

If $C \in \Psi$, the requirement is satisfied taking $\tilde{C} = C$. Consider now the situation $C \notin \Psi$ and suppose that, given the array of contracts offered at equilibrium, one has $U(C, e = H) > U(K, e = L)$, where K is the aggregate allocation optimally chosen by the entrepreneur when $e = L$ is selected. We now show that any investor whose offer is accepted has an incentive to deviate from his equilibrium offer.

[13] One cannot however conclude that the choice of priority rules has no impact on equilibrium existence. Consider, as an example, the situation where the entrepreneur decides to first repay lender 2. In this case the deviation to (I_2', R_2') is profitable if $\pi_L G(I^m + I_2' + A) - I_2' > 0$, which is verified for $I_2' < \frac{\pi_L G A}{1-\pi_L G}\frac{G\Delta\pi}{B}$. One can directly check that, given (1), there is a non-empty interval of I_2' that simultaneously satisfy the former inequality together with (10). This guarantees that, if lender 2 is repaid first under default, the monopolistic allocation cannot be supported at equilibrium.

Let C_i be the equilibrium offer of an active lender i, and suppose he deviates offering $C_i' = (I_i+\epsilon, R_i+\frac{\epsilon}{\pi_H}+\epsilon^2)$, for some strictly positive number ϵ. The deviation is profitable if the offer C_i' is accepted and the effort $e=H$ is selected. Indeed we have that $\pi_H(R_i+\frac{\epsilon}{\pi_H}+\epsilon^2)-I_i-\epsilon > \pi_H R_i - I_i$. Let us prove that for $\epsilon > 0$ small enough, the entrepreneur has an incentive to accept contract C_i' and choose $e=H$. To ease notations, denote $C=\sum C_j = \sum_{j\neq i} C_j + C_i$ and $C' = \sum_{j\neq i} C_j + C_i' = C + C_i' - C_i$. Write also $C'=(I',R')$ with $I'=I+\epsilon$ and $R' = R+\frac{\epsilon}{\pi_H}+\epsilon^2$. Now, since $C \in \mathcal{H}-\Psi$ we have:

$$\pi_H(G(I+A)-R) > \pi_L(G(I+A)-R) + B(I+A). \tag{12}$$

See that $C' \in \mathcal{H}-\Psi$ if and only if:

$$\pi_H(G(I+A)-R)+\pi_H(G\epsilon-\frac{\epsilon}{\pi_H}-\epsilon^2) > \pi_L(G(I+A)-R)+\pi_L(G\epsilon-\frac{\epsilon}{\pi_L}-\epsilon^2)+B(I+A+\epsilon). \tag{13}$$

Given (12), a sufficient condition for (13) to hold is:

$$\epsilon[\Delta\pi G - \epsilon - B] \geq 0.$$

Recalling that $\Delta\pi G - B > 0$, the above condition is true if ϵ is small enough, and we have $C' \in \mathcal{H}-\Psi$. We next show that C' is chosen by the entrepreneur. See that:

$$\begin{aligned}
U(C+C_i'-C_i, e=H) &= U(\sum_{i\neq j} C_i + C_i', e=H) \\
&= \pi_H(G(I+A)-R) - A + \pi_H(G(I_i'-I_i)-(R_i'-R_i)) \\
&= \pi_H(G(I+A)-R) - A + \pi_H(G\epsilon - \frac{\epsilon}{\pi_H} - \epsilon^2) \\
&> \pi_H(G(I+A)-R) - A
\end{aligned}$$

where the last inequality holds for ϵ sufficiently small since $\pi_H G > 1$. Therefore, $U(C', e=H) > U(C, e=H)$. Last, since $U(C, e=H) > U(K, e=L)$ it follows that for ϵ sufficiently small $U(C + C_i' - C_i, e = H) > U(K - C_i + C_i', e = L)$. Therefore the offer C_i' constitutes a profitable deviation for investor i, which contradicts the claim that C is an equilibrium allocation.

Proof of assertion (ii).

We proceed by way of contradiction. Suppose that, given the equilibrium offers, the borrower cannot achieve the allocation $\bar{C} \in \Psi$. In a first step we show that the set J of lenders who are inactive at equilibrium necessarily includes at least one lender j offering a contract $C_j = (I_j, R_j)$ of price $p_j \in (G, G + \frac{B}{\pi_L})$. In a second step, we show that lender j can profitably deviate inducing the entrepreneur to select $e = H$, which contradicts the fact that C is an equilibrium allocation. Formally, assertion (ii) of Proposition 2 results from the two following lemmas.

Lemma 2 *Assume $\bar{C}$ cannot be achieved. Let J be the set of investors who are inactive at equilibrium and consider $J_2 = \{j \in J \,:\, p_j \in (G, G + \frac{B}{\pi_L})\}$. Then, we have that $J_2 \neq \emptyset$.*

Lemma 3 *Assume $\bar{C}$ cannot be achieved and take any lender $s \in J_2$. Consider the contract $C_s' = (\epsilon, \frac{\epsilon}{\pi_H} + \epsilon^2)$ where ϵ is strictly positive and sufficiently small. C_s' is a profitable deviation for lender s.*

Proof of Lemma 2. Let J be the set of the investors who are inactive at equilibrium. By assertion (i), if $C \notin \Psi$, J is not empty, since the allocation $\tilde{C} \in \mathcal{L}$ such that $U(C, e = H) = U(\tilde{C}, e = L)$ must be available. In addition, it follows from lemma 1 that all contracts accepted when $e = H$ are also accepted when $e = L$. If J is empty, one has $C = \tilde{C}$ and in turn $C \in \Psi$, which contradicts $C \notin \Psi$. We also know from lemma 1 that each C_j contract, with $j \in J$, must be such that $p_j = \frac{R_j}{I_j} \geq G$. In particular, $J = J_1 \cup J_2 \cup J_3$ with:

1. $J_1 = \{j \in J : p_j = G\}$
2. $J_2 = \{j \in J : p_j \in (G, G + \frac{B}{\pi_L})\}$
3. $J_3 = \{j \in J : p_j \geq G + \frac{B}{\pi_L}\}$,

Thus $C + \sum_{j\in J_1} C_j + \sum_{j\in J_2} C_j$ is an allocation which utility is maximum in $\mathcal{L}$ and from (i) one gets that $C + \sum_{j\in J_1} C_j + \sum_{j\in J_2} C_j = \tilde{C}$ (by lemma 1, a contract such that $p_i < G + \frac{B}{\pi_L}$ is always accepted by the entrepreneur with effort $e = L$). In words, to achieve the allocation $\tilde{C}$ all contracts offered at a price smaller than $G + \frac{B}{\pi_L}$ must be accepted. It follows that if $\bar{C}$ cannot be achieved, then J_2 is necessarily non-empty. By way of contradiction, assume that J_2 is empty, then $\tilde{C} = C + \sum_{j\in J_1} C_j$. By (i), we have: $U(C, e = H) = U(\tilde{C}, e = L)$. But we also have $U(C, e = H) = U(C + J_1, e = H) = U(\tilde{C}, e = H)$. This implies that $\tilde{C} = \bar{C}$, which yields a contradiction.

Proof of Lemma 3. Let us consider a investor $s \in J_2$ and suppose he deviates and offers $C'_s = (\epsilon, \frac{\epsilon}{\pi_H} + \epsilon^2)$ for some strictly positive ϵ. The deviation has been constructed to be profitable if the offer C'_s is accepted and the effort $e = H$ is selected by the entrepreneur. First see that the entrepreneur accepts contract C'_s. Indeed,

$$U(C + C'_s, e = H) > U(C, e = H) \Leftrightarrow G > \frac{1}{\pi_H} + \epsilon,$$

which holds for ϵ not too large. We next show that the entrepreneur prefers to select $e = H$ given the deviation C'_s.

Consider the limit case $\epsilon = 0$ that is $C'_s = (0, 0)$. This means that investor s removes his initial offer. Recall that $\tilde{C} = C + \sum_{j\in J_1} C_j + \sum_{j\in J_2} C_j$. We thus have:

$$U(C + \sum_{j\in J_1} C_j + \sum_{j\in J_2, j\neq s} C_j, e = L) < U(\tilde{C}, e = L).$$

That is, if investor s withdraws his offer, the allocation $\tilde{C}$ is no more available for the entrepreneur. If, on the contrary, the entrepreneur chooses $e = H$, she can still achieve her equilibrium payoff

$U(C, e = H)$. By continuity, for $\epsilon > 0$ small, when investor s offers $C'_s = (\epsilon, \frac{\epsilon}{\pi_H} + \epsilon^2)$, we have:

$$U(C + C'_s + \sum_{j \in J_1} C_j + \sum_{j \in J_2, j \neq s} C_j, e = L) < U(\tilde{C}, e = L) = U(C, e = H).$$

Recall that $U(C + C'_s, e = H) > U(C, H)$ to establish:

$$U(C + C'_s, e = H) > U(C + C'_s + \sum_{j \in J_1} C_j + \sum_{j \in J_2, j \neq s} C_j, e = L) > U(C + C'_s, e = L),$$

where the last inequality comes from the fact that contracts in J_1 and J_2 are always accepted when $e = L$. In words, following a unilateral deviation to C'_s, the entrepreneur has an incentive to trade the deviating contract and to select $e = H$, which guarantees that the deviation is indeed profitable.

A consequence of properties i) and ii) is that J_2 is empty. If not, we cannot have $U(C, e = H) = U(\bar{C}, e = H) = U(\tilde{C}, e = L)$.

■

Proof of Corollary 1.

The proof is developed by contradiction. Given the equilibrium offers' array $\{C_1, \cdots, C_N\}$, let $C \in H - \Psi$ be the corresponding aggregate allocation supported at equilibrium. It follows from proposition 2 that at least one lender will not be trading at equilibrium. In addition, it follows from lemma 3 that $J_2 = \emptyset$. Since the allocation $\bar{C} \in \Psi$ must be available to the borrower at equilibrium, one necessarily has: $\bar{C} = C + \sum_{j \in J_1} C_j$. Consider now any lender $s \in J_1$, and suppose he deviates towards the alternative offer $C'_s = (\epsilon, \dfrac{\epsilon}{\pi_H} + \epsilon^2)$ for some strictly positive ϵ. The deviation has been constructed to be profitable if the offer C'_s is accepted and the effort $e = H$ is selected. Following the same reasoning developed in Lemma 3 one can show that there is indeed a profitable deviation.

■

Proof of Proposition 3

The proof is developed by contradiction. Let C_i be a contract of price $p_i < G$ which is traded at equilibrium. This indeed guarantees that the entrepreneur's payoff is strictly greater than her reservation utility $U(0)$. We now show that an active investor proposing C_i can profitably deviate and reduce the entrepreneur's utility. Suppose that investor i deviates to $C_i' = \left(I_i - \epsilon, R_i - \epsilon(G - \frac{B}{\Delta\pi}) - \epsilon^2\right)$ for some strictly positive ϵ. Under assumption (2), if the deviation is accepted and the entrepreneur chooses $e = H$, such a deviation increases investor i's profit for ϵ sufficiently small. We check below that the deviation is accepted and the entrepreneur chooses $e = H$.

By Lemma 1, the entrepreneur always has an incentive to trade the C_i' contract if $p(C_i') < G$. This is equivalent to

$$
\begin{aligned}
& R_i - \epsilon\left(G - \tfrac{B}{\Delta\pi}\right) - \epsilon^2 < G(I_i - \epsilon) \\
\Leftrightarrow \qquad & R_i - GI_i + \epsilon\left(\tfrac{B}{\Delta\pi} - \epsilon\right) < 0
\end{aligned}
$$

Since $R_i - GI_i < 0$, the above condition holds when ϵ is small enough.

Next, we show that in every continuation game following the deviation to C_i' the entrepreneur selects $e = H$.

Given the offers $\{C_1, \cdots, C_i', \cdots C_N\}$, we take $\mathcal{I} = \{i \in N \backslash J_1\}$ to be the set of active investors at the equilibrium.

Then, the agent maximizes his utility in $\mathcal{L}$ when buying all contract of price less than G, plus all contracts of price G. Indeed, by Proposition 2, J_2 is empty: No contract can be offered with $p \in (G, G + \frac{B}{\pi_L})$. The set of such contracts is

$$\sum_{i \in \mathcal{I}} C_i + \sum_{j \in J_1} C_j = \bar{C} - C_i + C_i'.$$

See that

$$
\begin{aligned}
& U(\bar{C}-C_i+C_i', e=H) > U(\bar{C}-C_i+C_i', e=L) \\
\Leftrightarrow \quad & \pi_H\left\{G(\bar{I}-\epsilon)-(\bar{R}-\epsilon(G-\tfrac{B}{\Delta\pi})-\epsilon^2)\right\} > \pi_L\left\{G(\bar{I}-\epsilon)-(\bar{R}-\epsilon(G-\tfrac{B}{\Delta\pi})-\epsilon^2)\right\}+B(\bar{I}-\epsilon) \\
\Leftrightarrow \quad & \epsilon^2 > 0.
\end{aligned}
$$

The entrepreneur's maximal utility when $e = L$ is strictly lower than his utility when $e = H$. Any other feasible allocation, by adding contracts of J_3 and by subtracting contracts in J_1 or in $\mathcal{I}$ reduces the entrepreneur's utility. Therefore, following the deviation C_i', the entrepreneur strictly prefers to choose $e = H$: No contract with $p < G$ can be offered at equilibrium.

■

Proof of Proposition 4.

Let $C = (I, R) \in \mathcal{H}$ be a feasible aggregate allocation. Define $\mathring{I} \in \mathbb{R}_+$ as the investment level such that

$$\pi_H[G(I+A)-R] = B\big(I+\mathring{I}+A\big). \tag{14}$$

According to equation (14), the entrepreneur obtains the equilibrium utility if she borrows $\mathring{I}$ in addition to the equilibrium investment I, chooses $e = L$ and is left with his private benefit only. Recall that, to guarantee borrower's participation, it must be $R \leq GI$. It follows that

$$B\,\mathring{I} = \pi_H[G(I+A)-R] - B\,(I+A) \geq \pi_L[G(I+A)-R] \geq \pi_L GA > 0 \;\; \forall (I,R) \in \mathcal{H},$$

which guarantees $\mathring{I} > 0$. Throughout the proof of Proposition 4 we shall use repeatedly the above relation (14) together with the following Lemma.

Lemma 4 *Let us consider the function f defined on $[\pi_H GA, U^c]$ by the relation $f(U) = U^c - BI^c - U$, where $U^c = U(I^c, R^c, H)$ denotes the entrepreneur's expected utility under high effort obtained with the competitive allocation (I^c, R^c). The following holds* *(i) If* $B \leq \pi_H G - 1$*, then there exists a unique* $U_B^* \in [\pi_H GA, U^c]$ *such that* $f(U(I,R,H)) < 0$ *for any feasible allocation* $(I,R) \in \mathcal{H}$ *satisfying* $U(I,R,H) > U_B^*$*. Furthermore, any feasible allocation* $(I,R) \in \mathcal{H}$ *such that* $U(I,R,H) > U_B^*$ *satisfies the relation* $R < \frac{1}{\pi_H}(1+B)I$.

(ii) If $B > \pi_H G - 1$*, then* $f(U(I,R,H)) < 0$ *for any feasible allocation* $(I,R) \in \mathcal{H}$.

Observe that for any feasible allocation (I,R) in $\mathcal{H}$ we have that $f(U(I,R,H)) = U^c - B(I^c + I + \mathring{\mathrm{I}} + A)$. The proof of the Lemma relies then on a straightforward computation. We show below that, (i) if $B \leq \pi_H G - 1$ then any aggregate feasible allocation $C = (I,R) \in \mathcal{H}$ such that $U(I,R,H) > U_B^*$ can be supported at the equilibrium, (ii) if $B > \pi_H G - 1$ then any aggregate feasible allocation $C = (I,R) \in \mathcal{H}$ can be supported at the equilibrium.

The equilibrium we construct involves a large number of both active and inactive lenders. Consider the following profile of strategies. Each lender $i = 1, 2, ..., M$ offers:

$$\begin{cases} (\frac{I}{M}, \frac{R}{M}) & \text{if} \quad \text{the total investment is } I \\ (\frac{I}{M}, \frac{\mathring{\mathrm{R}}}{M-1}) & \text{if} \quad \text{the total investment is } \frac{(M-1)I}{M} \\ (\frac{I}{M}, G(\hat{I} + A)) & \text{if} \quad \text{the total investment is } \hat{I} \notin \{\frac{(M-1)I}{M}, I\} \end{cases} \tag{15}$$

where $\mathring{\mathrm{R}}$ is defined by:

$$\pi_H[G(I+A) - R] = \pi_H[G(\frac{M-1}{M}I + A) - \mathring{\mathrm{R}}], \tag{16}$$

which implies $\mathring{\mathrm{R}} = R - \frac{GI}{M}$.

Each lender $j = M+1, \ldots, N$ proposes the pair $\left(\frac{\mathring{\mathrm{I}}}{N-M}, G(\hat{I} + A)\right)$ $\forall \hat{I} \in \mathbb{R}_+$.

Let us show that these strategies constitute an equilibrium in which the entrepreneur accepts the offers of investors $1, .., M$, rejects those of investor $M+1, \ldots, N$ and selects $e = H$. Therefore, only M investors are active. If $B \leq \pi_H G - 1$ we restrict the feasible aggregate allocations $(I,R) \in \mathcal{H}$ to those satisfying $U(I,R,H) > U_B^*$. If $B > \pi_H G - 1$, (I,R) represents any feasible allocation in $\mathcal{H}$. We shall assume that the number of investors N is sufficiently large. As we shall see the borrower's equilibrium strategy is constructed in such a way that, if default takes place, she first repays all non-deviating investors. We shall also see that a situation in which lenders are repaid to some pro-rata rules is consistent with the above borrower's equilibrium strategy.

1. We first show that given the investors' offers, it is a best reply for the entrepreneur to accept the proposals of investors $1, \ldots, M$ and to reject those of investors $M+1, \ldots, N$. Following the equilibrium strategy, the entrepreneur obtains the payoff $\pi_H[G(I+A)-R] \geq \pi_H GA$. See that no other portfolio choice associated to $e=H$ provides the entrepreneur with a strictly higher payoff: indeed, accepting $M-1$ contracts yields the same utility for the entrepreneur. Accepting any of the $N-M$ contracts triggers $e=L$. Accepting less than $M-1$ contracts also triggers $e=L$. Next, see that the entrepreneur cannot increase her utility by choosing $e=L$ and accepting all N contracts. She would then obtain an aggregate loan of $I+\overset{\circ}{\mathrm{I}}$ and a utility level of $B\big(I+\overset{\circ}{\mathrm{I}}+A\big) = \pi_H[G(I+A)-R]$, by (14).

2. We next show that given the equilibrium strategies, none of the inactive investors $M+1, \ldots, N$ can profitably deviate.

• Consider an inactive investor j. Let us establish that he cannot propose a deviation (I'_j, R'_j) such that only his offer is accepted out of equilibrium. Define (i, r) as the investment-repayment pair that lies at the intersection between the zero-profit line (of investors) of slope $\frac{1}{\pi_H}$ and the entrepreneur's equilibrium indifference curve given (I, R). Check that $i = \dfrac{\pi_H(GI-R)}{\pi_H G-1}$ and $r = \dfrac{i}{\pi_H}$. Denote also (i_I, r_I) the intersection between the entrepreneur's indifference curve and Ψ, and the competitive allocation (I^c, R^c) defined by the intersection between the zero-profit line and Ψ. Any deviation (I'_j, R'_j) that provides a strictly positive profit to lender j and (weakly) increases the utility of the entrepreneur if only (I'_j, R'_j) is accepted must lie in the triangle defined by (i, r), (i_I, r_I) and (I^c, R^c). We show below that any such deviation induces the entrepreneur to accept all contracts and exert $e=L$ if N is sufficiently large. To do so, we prove that, if N is large enough then the function

$$F(I'_j, R'_j; I, R) = \pi_H \left(G(I'_j + A) - R'_j\right) - B \left(I'_j + I + \mathring{I} \, \frac{N-M-1}{N-M} + A\right)$$

is negative at (i, r). This will in turn imply that F is negative at any point in the triangle $(i, r), (i_I, r_I), (I^c, R^c)$ which corresponds to the set of admissible deviations (I'_j, R'_j). Consider therefore that investor j deviates and proposes the pair (i, r), the entrepreneur prefers to choose $e = L$ if:

$$\pi_H \left(G(i + A) - r\right) - B \left(i + I + \mathring{I} \, \tfrac{N-M-1}{N-M} + A\right) \leq 0$$
$$\iff \quad B\left(i - \frac{\mathring{I}}{N-M}\right) \geq 0. \tag{17}$$

We use then the definitions of i and $\mathring{I}$ to rewrite (17) as:

$$(N-M)\frac{\pi_H(GI - R)}{\pi_H G - 1} - \pi_H \frac{G(I + A) - R}{B} + (I + A) \geq 0. \tag{18}$$

If N is large enough, then (18) is satisfied for all feasible allocations (I, R): F is negative at the point (i, r). Next, see that for a given I'_j, the minimum value of R'_j such that the deviation is in the admissible triangle $(i, r), (i_I, r_I), (I^c, R^c)$ is defined by $R'_j = \frac{I'_j}{\pi_H}$ and observe that

$$\left.\frac{dF}{dI'_j}\right|_{R'_j = \frac{I'_j}{\pi_H}} = \pi_H G - 1 - B. \tag{19}$$

First consider the case where the private benefit B satisfies $B \leq \pi_H G - 1$ then the function $F(I'_j, \frac{I'_j}{\pi_H}; I, R)$ increases with I'_j and reaches its maximum at $I'_j = I^c$. Because we assume in this case that the allocation (I, R) satisfies $U(I, R, H) \geq U_B^*$ this maximum is negative from Lemma 4. This yields $F(I'_j, R'_j; I, R) < 0$ for any pair $(I'_j, \frac{I'_j}{\pi_H})$ with $I'_j \in [i, I^c]$. Finally, since F is decreasing in R'_j one gets that F is negative at any point in the triangle $(i, r), (i_I, r_I), (I^c, R^c)$ satisfying $U(I, R, H) \geq U_B^*$. Thus, for every feasible allocation (I, R) with $U(I, R, H) \geq U_B^*$ there always exists a sufficiently high number N of investors such that none of the inactive investors has a unilateral incentive to deviate as long as the entrepreneur accepts only his offer out of equilibrium. If the private benefit B is such that $B > \pi_H G - 1$ then the function $F(I'_j, \frac{I'_j}{\pi_H}; I, R)$ decreases with I'_j and the conclusion clearly holds for any feasible allocation (I, R) in the triangle $(i, r), (i_I, r_I), (I^c, R^c)$.

• Let us now establish that no inactive investor j has an incentive to deviate so that the entrepreneur trades several contracts out of equilibrium. For a deviation to be profitable, the entrepreneur must choose $e = H$ following the deviation. This is because the deviating investor is repaid after the others in case of default. Observe that the entrepreneur has an incentive to select $e = H$ only if the investment she trades in the aggregate is either I, or $\frac{(M-1)I}{M}$. In all other cases, the entrepreneur optimally chooses $e = L$ since all the cash-flow is paid to investors. It follows that only a subset of the active investors' offers must be traded out of equilibrium (and no offer from passive investors). Let (I'_j, R'_j) be a unilateral deviation by investor j, and let $m < M$ be the number of additional contracts which are traded following the deviation.

We first consider the case where the aggregate level of investment is equal to I; one we must have:

$$I = I'_j + \frac{m}{M} I \iff I'_j = \frac{M-m}{M} I \tag{20}$$

The borrower will accept the deviating contract (I'_j, R'_j) and choose $e = H$ if

$$\pi_H \left[G(I + A) - R'_j - \frac{m}{M} R \right] > B \left(I + \mathring{I} + A + I'_j - \frac{\mathring{I}}{N - M} \right), \tag{21}$$

which, given (14) and (20), corresponds to

$$\begin{aligned} &\pi_H \left[(1 - \frac{m}{M}) R - R'_j \right] > B \left(\frac{M-m}{M} I - \frac{\mathring{I}}{N - M} \right) \\ \Leftrightarrow \quad & B \frac{\mathring{I}}{N - M} > \frac{M-m}{M} \left(BI - \pi_H R + \pi_H R'_j \frac{M}{M-m} \right). \end{aligned} \tag{22}$$

For the deviation (I'_j, R'_j) to be profitable for the lender one must have $R'_j \pi_H > I'_j$. Then, given

(22), we have

$$B\frac{\mathring{\mathrm{I}}}{N-M} > \frac{M-m}{M}\left(BI - \pi_H R + I'_j \frac{M}{M-m}\right)$$

which, using (20), yields

$$B\frac{\mathring{\mathrm{I}}}{N-M} > \frac{M-m}{M}\left(I(1+B) - \pi_H R\right). \tag{23}$$

Assume that $B \leq \pi_H G - 1$. In this case the feasible allocation $(I, R) \in \mathcal{H}$ that we consider satisfies $U(I, R, H) > U_B^*$ which implies from Lemma 4 that $R < \frac{1}{\pi_H}(1+B)I$. It follows that condition (23) does not hold if N is high enough. This in turn contradicts (21), showing that there is no profitable deviation. If $B > \pi_H G - 1$, none of the feasible allocation $(I, R) \in \mathcal{H}$ satisfies (21).

We now consider the situation where the aggregate investment chosen at the deviation stage is equal to $\left(\frac{M-1}{M}\right) I$. The corresponding I'_j is such that:

$$\left(\frac{M-1}{M}\right) I = I'_j + \frac{m}{M} I \iff I'_j = \left(\frac{M-m-1}{M}\right) I \tag{24}$$

with $m < M-1$. A necessary condition for the deviation to (I'_j, R'_j) to be profitable is

$$\pi_H \left[G\left(\frac{M-1}{M} I + A\right) - R'_j - \frac{m}{M-1}\,\mathring{\mathrm{R}}\right] > B\left(I + \mathring{\mathrm{I}} + A + I'_j - \frac{\mathring{\mathrm{I}}}{N-M}\right), \tag{25}$$

which, given (14), corresponds to

$$B\frac{\mathring{\mathrm{I}}}{N-M} > BI'_j - \pi_H \frac{M-m-1}{M-1}\left(R - \frac{GI}{M}\right) + \pi_H R'_j \tag{26}$$

Using $\pi_H R'_j > I'_j$ and (24) together with the fact that $\frac{M-m-1}{M-1}\left(R - \frac{GI}{M}\right) \leq \frac{M-m-1}{M} GI$ because $R \leq GI$, we deduce from (26) that

$$B\frac{\mathring{\mathrm{I}}}{N-M} > \frac{M-m-1}{M} I(1 + B - \pi_H G). \tag{27}$$

As previously, if $B \leq \pi_H G - 1$, the inequality $R < \frac{1}{\pi_H}(1+B)I$ holds because we focus on allocation (I, R) such that $U(I, R, H) > U_H^*$. It follows that the (RHS) of (27) is positive. This implies that if N is large enough (25) is violated and there is no profitable deviation. If $B > \pi_H G - 1$, (25) is violated for any feasible allocation $(I, R) \in \mathcal{H}$. This concludes the analysis of deviations by passive investors.

3. We now turn to the proof that no active investor $(1, \ldots, M)$ can profitably deviate. Consider any of the investors who is active at equilibrium, say the k-th one.

• Consider first the situation where only the offer (I'_k, R'_k) of investor k is accepted out of equilibrium. Let us reformulate (i, r) as the investment-repayment pair that lies at the intersection between investor k's profit line and the entrepreneur's equilibrium indifference curve. Similarly, define (i_I, r_I) the intersection between the entrepreneur's indifference curve and Ψ, and (I^c, R^c) the competitive allocation defined by the intersection between investor k's profit line and Ψ. See that:

$$\begin{cases} \pi_H \left[G(I + A) - R\right] = \pi_H \left[G(i + A) - r\right] \\ \pi_H r - i = \pi_H \dfrac{R}{M} - \dfrac{I}{M} \end{cases} \tag{28}$$

It follows that $i = \dfrac{GI - R + (1/M)(R - I/\pi_H)}{G - 1/\pi_H}$. As before, we need to show that the function $F(I'_k, R'_k; I, R) = \pi_H \left(G(I'_k + A) - R'_k\right) - B\left(I'_k + I + \mathring{\mathrm{I}} - \frac{I}{M} + A\right)$ is negative at each point in the triangle $(i, r), (i_I, r_I), (I^c, R^c)$. To prove this, we first show that F is negative at (i, r). Assume that investor k deviates and offers (i, r), the entrepreneur prefers to choose $e = L$ if:

$$\pi_H \left(G(i + A) - r\right) - B\left(I + \mathring{\mathrm{I}} + A + i - \frac{I}{M}\right) = -B\left(i - \frac{I}{M}\right) \leq 0. \tag{29}$$

Given the definition of i, (29) can be rewritten as:

$$\frac{(GI - R)\left(1 - \frac{1}{M}\right)}{G - \frac{1}{\pi_H}} \geq 0, \tag{30}$$

which is always satisfied since $GI \geq R$.

Next, see that for a given I'_k, the minimum value of R'_k such that the deviation is in the admissible triangle $(i,r),(i_I,r_I),(i_P,r_P)$ is defined by $R'_k = \frac{I'_k+V}{\pi_H}$, where V represents the equilibrium profit of investor k. See that:

$$\left.\frac{dF}{dI'_k}\right|_{R'_k=\frac{I'_k+V}{\pi_H}} = \pi_H G - 1 - B. \tag{31}$$

If $B < \pi_H G - 1$ then the function $F(I'_k, \frac{I'_k+V}{\pi_H}; I, R)$ increases with I'_k and reaches its maximum at $I'_k = I^c$. Because the allocation (I,R) is such that $U(I,R,H) > U^*_B$ we obtain from Lemma 4 that this maximum is negative.[14] This yields that $F(I'_k, R'_k; I, R) < 0$ for any pair $(I'_k, \frac{I'_k+V}{\pi_H})$ with $I'_k > i$. Note also that F is decreasing in R'_k. This yields that F is negative at any point (I,R) in the triangle $(i,r),(i_I,r_I),(I^c,R^c)$ such that $U(I,R,H) > U^*_B$. It follows that none of the active investors has a unilateral incentive to deviate if the entrepreneur accepts only his offer out of equilibrium. Here again if $B > \pi_H G - 1$ then it is no more necessary to impose the condition $U(I,R,H) > U^*_B$ to get the result.

• Consider next the situation where an active investor k deviates and the entrepreneur trades several contracts out of equilibrium. As before, for a deviation to be profitable, default should necessarily be avoided. This can only happen if the aggregate investment selected by the entrepreneur at the deviation stage is either I, or $\frac{(M-1)I}{M}$.

We first consider the situation where such aggregate investment is set equal to I. Let m be the number of additional contracts which are traded following the deviation, we have: $I'_k = \dfrac{M-m}{M} I$.

A necessary condition for the deviation to be profitable is

$$\pi_H \left[G(I+A) - R'_k - \frac{m}{M} R \right] > B(I + \mathring{\text{I}} + A + I'_k - \frac{I}{M})$$

or equivalently using (14),

$$\pi_H \left[-R'_k + \frac{M-m}{M} R \right] > B \frac{M-m-1}{M} I. \tag{32}$$

[14] Very precisely, this easy computation also uses the relation $BI < V$ that follows from the inequality $\pi_H R > I$ together with the inequality $B < 1$ that is implied by relation (1).

For the deviation (I'_k, R'_k) to be profitable for the lender one must have $\pi_H R'_k - I'_k > \pi_H \frac{R}{M} - \frac{I}{M}$. It follows that, if (32) holds then, necessarily

$$\frac{M-m-1}{M}\pi_H R > \frac{M-m-1}{M}(B+1)I. \tag{33}$$

Again, if $B \leq \pi_H G - 1$, then the allocations (I, R) that we consider satisfy $U(I, R, H) > U_H^*$. From Lemma 4 this implies that $\pi_H R < (B+1)I$. If $B \leq \pi_H G - 1$, the inequality $\pi_H R < (B+1)I$ is always satisfied for any feasible allocation $(I, R) \in \mathcal{H}$. This contradicts condition (33).

Finally, if the aggregate investment chosen at the deviation stage is equal to $\left(\frac{M-1}{M}\right) I$, we have $I'_k = \frac{M-m-1}{M} I$. A necessary condition for the deviation to be profitable is

$$\pi_H \left[G\left(I + A - \frac{I}{M}\right) - R'_k - \frac{m}{M-1}\,\mathring{\mathrm{R}} \right] > B(I + \mathring{\mathrm{I}} + A + I'_k - \frac{I}{M}),$$

equivalently, using (14) and the definition of $\mathring{\mathrm{R}}$,

$$\pi_H \left[-\left(M-m-1\right) \frac{GI}{M\left(M-1\right)} + \frac{M-m-1}{M-1} R - R'_k \right] > BI\left(\frac{M-m-2}{M}\right) \tag{34}$$

The deviation (I'_k, R'_k) which must be profitable for the lender satisfies $\pi_H R'_k - I'_k > \pi_H \frac{\mathring{\mathrm{R}}}{M-1} - \frac{I}{M}$. It follows that if (34) holds then necessarily,

$$\pi_H \left[R - \frac{1}{M} GI \right] \frac{M-m-2}{M-1} > (B+1)I\left(\frac{M-m-2}{M}\right). \tag{35}$$

Observe that the (LHS) of (35) is lower than $\pi_H \frac{M-m-2}{M} GI$, so that given $\pi_H R < I(1+B)$, the inequality (34) cannot be satisfied.

This completes the proof of Proposition 4. When $B < \pi_H G - 1$, any allocation $C = (I, R) \in \mathcal{H}$ such that $R \in [\frac{1}{\pi_H} I, GI]$ with $U(I, R, H) > U_B^*$ can be supported at equilibrium. When $B < \pi_H G - 1$ any allocation $C = (I, R) \in \mathcal{H}$ can be supported at the equilibrium.

Remark: Proposition 4 also holds if defaulted bonds are repaid *pro rata*. To prove this let us first consider an inactive lender j who deviates and thus proposes (I'_j, R'_j) such that the borrower earns a utility strictly greater than the equilibrium one by selecting $e = L$ accepting all contracts and defaulting even in the success state. That is,

$$B(I'_j + I + \frac{N-M-1}{N-M} \mathring{I} + A) > B(I + \mathring{I} + A)$$

or equivalently, using (14)

$$I'_j > \frac{\mathring{I}}{N-M}. \tag{36}$$

In a *pro rata* environment, the deviation is profitable for the lender iff

$$\begin{aligned} &\pi_L G \left(I'_j + I + \frac{N-M-1}{N-M} \mathring{I} + A \right) \frac{I'_j}{I'_j + I + \frac{N-M-1}{N-M} \mathring{I}} > I'_j \\ &\iff (\pi_L G - 1) \left(I + I'_j + \frac{N-M-1}{N-M} \mathring{I} \right) + \pi_L G A > 0. \end{aligned} \tag{37}$$

Relation (36) together with $B \mathring{I} \geq \pi_L G A$ implies

$$\begin{aligned} & (\pi_L G - 1) \left(I + I'_j + \frac{N-M-1}{N-M} \mathring{I} \right) + \pi_L G A \\ < \; & (\pi_L G - 1) \left(I + \frac{1}{N-M} \frac{1}{B} \pi_L G A + \frac{N-M-1}{N-M} \frac{1}{B} \pi_L G A \right) + \pi_L G A \\ = \; & (\pi_L G - 1) I + \pi_L G A \left(\frac{1}{B} (\pi_L G - 1) + 1 \right) < 0 \end{aligned}$$

where the last inequality comes from (1) and (2). This shows that relations (36) and (37) are not compatible.

Second let consider an active lender j who deviates and proposes (I'_j, R'_j). Observe that, necessarily, $I'_j > I_j = \frac{I}{M}$. Indeed, if it was not the case, because of Equation (14), the borrower's utility at the deviation stage would be lower than her equilibrium utility. Formally,

$$B \left(I'_j + \frac{M-1}{M} I + \mathring{I} + A \right) < B(I + \mathring{I} + A) = \pi_H (G(I + A) - R).$$

We thus have that $I'_J > \frac{I}{M}$. A direct computation that uses Equation (1) shows that relation $I'_J > \frac{I}{M}$ cannot hold together with inequality

$$\pi_L G\left(I'_j + \frac{M-1}{M} I + \mathring{I} + A\right) \frac{I'_j}{I'_j + \frac{M-1}{M} I + \mathring{I}} > I'_j. \tag{38}$$

This means that the deviation is not profitable for the lender j under a *pro rata* rule.

$\blacksquare$

Proof of Proposition 5.

The proof follows the standard logic of price competition. Let $C \neq C^c$ be a equilibrium allocation where the entrepreneur selects $e = H$. Consider first the situation where investors are earning a strictly positive profit at equilibrium, which implies that $R > \frac{1}{\pi_H} I$. Take any of the investors earning the smallest profit at equilibrium (the same logic goes through if one considers a deviation by a passive investor), say the i-th one, and suppose he deviates proposing:

$$\begin{cases} (I'_i, R'_i) = (I, R - \varepsilon) & \text{if the aggregate level of investment is } I \\ (I'_i, R'_i) = (I, G(2I - I_i + \mathring{I} + A) & \text{if the aggregate level of investment is } \neq I, \end{cases}$$

where ϵ has been taken to be small enough. The deviation guarantees that investor i earns (almost) the aggregate equilibrium payoff. The entrepreneur has a clear incentive to accept the offer, since she can achieve a payoff strictly greater than the equilibrium one, and to accept only investor i's offer: any alternative choice would indeed induce her to reject the offer of investor i (because strategic default is ruled out). Also, the entrepreneur clearly selects $e = H$ at the deviation stage, which guarantees that the deviation is profitable in the first place.

If aggregate profits are zero at equilibrium, i.e. if $R = \frac{1}{\pi_H} I$, it is always possible for any of the investors, say the i-th one, to profitably deviate proposing:

$$\begin{cases} (I'_i, R'_i) = (I + \epsilon, R + G\epsilon^2) & \text{if the aggregate level of investment is } I \\ (I'_i, R'_i) = (I + \epsilon, G(2I + \epsilon - I_i + \mathring{I} + A)) & \text{if the aggregate level of investment is } \neq I, \end{cases}$$

where ϵ has been taken to be small enough. The deviation guarantees that investor i earns a strictly positive profit if $e = H$ is chosen. The entrepreneur has a clear incentive to accept the offer, since she can achieve a payoff strictly greater than the equilibrium one. In addition, the optimal level of investment she will be trading at the deviation stage is exactly I_i': any alternative choice would indeed induce her to reject the offer of investor i. It follows that $e = H$ is selected at the deviation stage, which guarantees that the deviation is profitable in the first place. Assertion (i) is proven. Assertion (ii) directly follows from the proof of Proposition 1.

■

References

ATTAR, A., E. CAMPIONI, AND G. PIASER (2006): "Multiple lending and constrained efficiency in the credit market," *Contributions to Theoretical Economics*, 6(1).

ATTAR, A., AND A. CHASSAGNON (2009): "On moral hazard and nonexclusive contracts," *Journal of Mathematical Economics*, forthcoming.

BENNARDO, A., M. PAGANO, AND S. PICCOLO (2009): "Multiple-Bank Lending, Creditor Rights and Information Sharing," University of Naples Federico II, mimeo.

BISIN, A., AND D. GUAITOLI (2004): "Moral Hazard and Nonexclusive Contracts," *RAND Journal of Economics*, 35(2), 306–328.

BISIN, A., AND A. RAMPINI (2006): "Exclusive contracts and the Institution of Bankruptcy," *Economic Theory*, 27, 277–304.

BIZER, D., AND P. DEMARZO (1992): "Sequential Banking," *Journal of Political Economy*, 100(1), 41–61.

DEMIROGLU, C., AND C. JAMES (2008): "The information content of bank loan covenants," http://ssrn.com/abstract=959393, mimeo.

HOLMSTROM, B., AND J. TIROLE (1997): "Financial Intermediation, Loanable Funds and the Real Sector," *Quarterly Journal of Economics*, 112(3), 663–691.

——— (1998): "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106(1), 1–40.

PAGLIA, J., AND D. MULLINEAUX (2006): "An empirical exploration of financial covenants in large bank loans," *Banks and Bank Systems*, 1(2), 103–122.

PARLOUR, C. A., AND U. RAJAN (2001): "Competition in Loan Contracts," *American Economic Review*, 91(5), 1311–1328.

TIROLE, J. (2006): *The Theory of Corporate Finance*. Princeton University Press.