Thresholding Events of Extreme in Simultaneous Monitoring of Multiple Risks

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SUPPLEMENTAL MATERIALS

Proof of Theorem 6.1. For simplicity we write $\hat{a} = \hat{a}_{n/k}, \hat{b} = \hat{b}_{n/k}, a = a_{n/k}, b = b_{n/k}$ and $\hat{\gamma} = \hat{\gamma}_n$. Observe that $d_n \to \infty$. We first show that for $-\frac{1}{2} < \gamma < 0$,

$$\mathbb{P}\left(\frac{\hat{b}-b}{a} + \frac{\hat{a}}{a}\frac{d_n^{\hat{\gamma}}-1}{\hat{\gamma}} < -\frac{1}{\gamma}\right) \to 0 \quad (n \to \infty). \tag{7.1}$$

Note that, the condition (e) implies

$$1 + \frac{\hat{\gamma}\left(-\frac{1}{\gamma} - \frac{B_n}{\sqrt{k}}\right)}{1 + \frac{A_n}{\sqrt{k}}} = O_p\left(\frac{1}{\sqrt{k}}\right). \tag{7.2}$$

Applying condition (c) twice and also (e), we obtain

$$d_n^{\hat{\gamma}}\sqrt{k} = d_n^{\gamma}\sqrt{k}e^{\frac{\log d_n}{\sqrt{k}}\cdot\Gamma_n}, \tag{7.3}$$

which tends to infinity in probability. Combination of (7.2) and (7.3) easily yields (7.1).

Using condition (d), it follows from Lemma 2.4.1 in Li (2004) in conjunction with (7.1) that for all $\gamma > -\frac{1}{2}$

$$\frac{\bar{F}(\hat{x}_p)}{p} = d_n\left(1 + \gamma\left(\frac{\hat{b}-b}{a} + \frac{\hat{a}}{a}\frac{d_n^{\hat{\gamma}}-1}{\hat{\gamma}}\right)\right)^{-\frac{1}{\gamma}}(1 + o_p(1)).$$

Therefore it remains to show that

$$d_n\left(1 + \gamma\left(\frac{\hat{b}-b}{a} + \frac{\hat{a}}{a}\frac{d_n^{\hat{\gamma}}-1}{\hat{\gamma}}\right)\right)^{-\frac{1}{\gamma}} \xrightarrow{p} 1. \tag{7.4}$$

Note that, as $x \to \infty$,

$$q_\gamma(x) \sim \begin{cases} \frac{1}{\gamma} x^\gamma \log x & \gamma > 0 \\ \frac{1}{2}(\log x)^2 & \gamma = 0 \\ \frac{1}{\gamma^2} & \gamma < 0 \end{cases}$$

Observe that condition (c) is equivalent to

$$\begin{cases} \frac{\log d_n}{\sqrt{k}} \to 0, & \text{for } \gamma > 0, \\ \frac{(\log d_n)^2}{\sqrt{k}} \to 0, & \text{for } \gamma = 0, \\ d_n^\gamma \sqrt{k} \to \infty, & \text{for } \gamma < 0. \end{cases} \tag{7.5}$$

Next we prove (7.4); part of the proof is similar to that of Proposition 8.2.9 in de Haan and Ferreira (2006). First, we take $\gamma \neq 0$:

$$\begin{aligned} & d_n \left(1 + \gamma \left(\frac{\hat{b} - b}{a} + \frac{\hat{a}}{a} \frac{d_n^{\hat{\gamma}} - 1}{\hat{\gamma}}\right)\right)^{-\frac{1}{\gamma}} \\ = \quad & d_n \left(1 + \gamma \frac{B_n}{\sqrt{k}} + \left(1 + \frac{A_n}{\sqrt{k}}\right)\left(1 - \frac{\Gamma_n}{\sqrt{k}\hat{\gamma}}\right)\left(d_n^{\hat{\gamma}} - 1\right)\right)^{-\frac{1}{\gamma}} \\ =: \quad & d_n \left(\left(1 + \frac{D_n}{\sqrt{k}}\right) d_n^{\hat{\gamma}} + \frac{E_n}{\sqrt{k}}\right)^{-\frac{1}{\gamma}} \\ = \quad & d_n^{1-\frac{\hat{\gamma}}{\gamma}} \left(1 + \frac{D_n}{\sqrt{k}} + \frac{E_n}{d_n^{\hat{\gamma}} \sqrt{k}}\right)^{-\frac{1}{\gamma}} \quad =: s_n. \end{aligned}$$

Note that D_n and E_n are $O_p(1)$ due to (e), and also that $d_n^{\hat{\gamma}} \sqrt{k} \to \infty$ (for $\gamma < 0$ see (7.3)). Hence, we have

$$\left(1 + \frac{D_n}{\sqrt{k}} + \frac{E_n}{d_n^{\hat{\gamma}} \sqrt{k}}\right)^{-\frac{1}{\gamma}} \xrightarrow{p} 1, \quad \text{and} \quad d_n^{1-\frac{\hat{\gamma}}{\gamma}} = e^{-\frac{\log d_n}{\sqrt{k}} \cdot \frac{\Gamma_n}{\gamma}} \xrightarrow{p} 1.$$

Consequently, $s_n \xrightarrow{p} 1$, and thus (7.4) holds for $\gamma \neq 0$.

For $\gamma = 0$, the proof of (7.4) goes as follows. By definition $(1 + \gamma x)^{-1/\gamma} = e^{-x}$ in this case. Note then

$$d_n \exp\left(-\frac{\hat{b} - b}{a} - \frac{\hat{a}}{a} \frac{d_n^{\hat{\gamma}} - 1}{\hat{\gamma}}\right) = \exp\left(-\frac{B_n}{\sqrt{k}} - \left(1 + \frac{A_n}{\sqrt{k}}\right)\left(\frac{d_n^{\hat{\gamma}} - 1}{\hat{\gamma}} - \log d_n\right) - \frac{A_n}{\sqrt{k}} \log d_n\right),$$

Clearly, if

$$\frac{d_n^{\hat{\gamma}} - 1}{\hat{\gamma}} - \log d_n \xrightarrow{p} 0, \tag{7.6}$$

holds, then (7.4) follows immediately from (e). To show (7.6), we observe that

$$\begin{aligned}\left|\frac{d_n^{\hat{\gamma}}-1}{\hat{\gamma}}-\log d_n\right| &= \left|\int_1^{d_n}\frac{s^{\hat{\gamma}}-1}{s}\,ds\right| = \left|\hat{\gamma}\int_1^{d_n}\int_1^s u^{\hat{\gamma}}\frac{1}{u}\,du\frac{1}{s}\,ds\right| \\ \leq\ & |\hat{\gamma}|\,d_n^{|\hat{\gamma}|}\log^2 d_n = \left|\hat{\gamma}\sqrt{k}\right|\,\frac{\log^2 d_n}{\sqrt{k}}\,e^{\frac{\log d_n}{\sqrt{k}}\left|\hat{\gamma}\sqrt{k}\right|}.\end{aligned}$$

The last expression tends to zero in probability under the conditions (c) and (e). This completes the proof.

Proof of Theorem 6.2. Note that

$$\frac{1}{p}P\left(\hat{Q}_p \triangle Q_p\right) \leq \frac{\left|\bar{F}_1(\hat{x}_{\hat{p}_1})-\bar{F}_1(x_{p_1})\right|}{p}+\frac{\left|\bar{F}_2(\hat{y}_{\hat{p}_2})-\bar{F}_2(y_{p_2})\right|}{p}.$$

We consider only the first term on the right, since the second can be handled similarly:

$$\frac{1}{p}\left|\bar{F}_1(\hat{x}_{\hat{p}_1})-\bar{F}_1(x_{p_1})\right| \leq \frac{1}{p}\left|\bar{F}_1(\hat{x}_{\hat{p}_1})-\bar{F}_1(x_{\hat{p}_1})\right|+\frac{1}{p}\left|\bar{F}_1(x_{\hat{p}_1})-\bar{F}_1(x_{p_1})\right|. \tag{7.7}$$

Note that the last term of (7.7) can be dealt with as follows:

$$\begin{aligned}\frac{1}{p}\left|\bar{F}_1(x_{\hat{p}_1})-\bar{F}_1(x_{p_1})\right| = \frac{1}{p}\left|\hat{p}_1-p_1\right| &\leq \frac{1}{p}\left|\frac{p}{\tilde{l}(1,c)}-\frac{p}{l(1,c)}\right|+\frac{1}{p}\left|\frac{p}{l(1,c)}-p_1\right| \\ &= \frac{\left|\tilde{l}(1,c)-l(1,c)\right|}{l(1,c)\tilde{l}(1,c)}+\left|\frac{1}{l(1,c)}-\frac{p_1}{p}\right|.\end{aligned} \tag{7.8}$$

Following Theorem 1 of Chapter 2 (consistency of $\hat{l}(x,y)$) in Huang (1992) or Theorem 7.2.1 in de Haan and Ferreira (2006), the first term of (7.8) tends to zero in probability. Since F is in the bivariate domain of attraction, by the argument given in Section 6.1 and the homogeneity of l listed in Section 3, the second term in (7.8) also tends to zero.

Thus, it remains to show that

$$\frac{1}{p}\left|\bar{F}_1(\hat{x}_{\hat{p}_1})-\bar{F}_1(x_{\hat{p}_1})\right| \xrightarrow{p} 0.$$

Note that since $p=\hat{p}_1\tilde{l}(1,c)$ and $\tilde{l}(1,c) \xrightarrow{p} l(1,c)$, it suffices to show

$$\frac{1}{\hat{p}_1}\left|\bar{F}_1(\hat{x}_{\hat{p}_1})-\bar{F}_1(x_{\hat{p}_1})\right| \xrightarrow{p} 0.$$

This clearly follows from the result in univariate case and Remark 6.2, since

$$\frac{\hat{p}_1}{p}=\frac{1}{\tilde{l}(1,c)} \xrightarrow{p} \frac{1}{l(1,c)} \in (0,\infty).$$

This completes the proof.

References

[1] de Haan, L. and Ferreira, A. (2006). *Extreme value theory: an introduction.* Springer, New York.

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[3] Li, D. (2004). *On extreme value approximation to tails of distribution functions.* Ph.D. thesis 342, Tinbergen Institute, Erasmus University, Rotterdam.