

Weighted Approximations of Tail Copula Processes with Application to Testing the Bivariate Extreme Value Condition

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APPENDIX

For the relative compactness, we need several lemmas. First we present in Lemma 4.1 sufficient conditions for relative compactness; the proof is similar to that of Theorem 15.5 in Billingsley (1968), see also Neuhaus (1971).

Lemma 4.1. *Let P_n be probability measures on (D_2, \mathcal{L}_d) . Suppose that, for each positive η , there exists an $M > 0$ such that*

$$P_n(x \in D_2 : |x(0, 0)| > M) \leq \eta, \quad n \geq 1.$$

Suppose further that, for each positive ε and η , there exist a δ , $0 < \delta < 1$, and an integer n_0 such that

$$P_n(x \in D_2 : \sup_{|u_1 - u_2| \leq \delta, |v_1 - v_2| \leq \delta} |x(u_1, v_1) - x(u_2, v_2)| > \varepsilon) \leq \eta, \quad n \geq n_0.$$

Then $\{P_n\}_{n \geq 1}$ is relatively compact.

Lemma 4.2. *Under the conditions of Proposition 4.1, for each $c, a > 0$*

(i) $\int_0^c \frac{W_{jn}(t)}{t} dt \sim N(0, \sigma_n^2)$, with $\sigma_n^2 \leq 2c$, $j = 1, 2$,

(ii) $P(\sup_{t \geq c} |\frac{W_{jn}(t)}{t}| \geq a) \leq 2P(|W(2/c)| \geq a)$, $j = 1, 2$, where W is a standard Wiener process.

Proof. (i) This follows from Proposition 1, page 42, in Shorack and Wellner (1986).

(ii) Let W be a standard Wiener process. Since $\{W(t)/t, t \geq c\} \stackrel{d}{=} \{W(1/t), t \geq c\}$, then

$$P(\sup_{t \geq c} |W(t)/t| \geq a) = P(\sup_{0 < s \leq 1/c} |W(s)| \geq a) \leq 2P(|W(1/c)| \geq a).$$

Write $\Lambda_{1n}(t)$ for $\Lambda_n([0, t] \times [0, \infty])$. Since $\{W_{1n}(t), t > 0\} \stackrel{d}{=} \{W(\Lambda_{1n}(t)), t > 0\}$, then

$$\begin{aligned} P(\sup_{t \geq c} |W_{1n}(t)/t| \geq a) &= P(\sup_{t \geq c} \left| \frac{W(\Lambda_{1n}(t)) \cdot \Lambda_{1n}(t)}{\Lambda_{1n}(t) \cdot t} \right| \geq a) \\ &\leq P(\sup_{\Lambda_{1n}(t) \geq c/2} |W(\Lambda_{1n}(t))/\Lambda_{1n}(t)| \geq a) \leq 2P(|W(2/c)| \geq a), \end{aligned}$$

eventually (since $t - 1/k \leq \Lambda_{1n}(t) \leq t$). For $j = 2$ the proof is the same. \square

Lemma 4.3. *Define*

$$H_n := \sup_{\theta \in [0, \pi/2]} |W_{\Lambda_n}(C_\theta) + Z_n(\theta)|.$$

Then under the conditions of Proposition 4.1, there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} P(H_n \geq a) = O(e^{-a}) \text{ as } a \rightarrow \infty.$$

Proof. Define $H_{1n} := \sup_{\theta \in [0, \pi/4]} |W_{\Lambda_n}(C_\theta) + Z_n(\theta)|$, $H_{2n} := \sup_{\theta \in (\pi/4, \pi/2)} |W_{\Lambda_n}(C_\theta) + Z_n(\theta)|$, and $H_{3n} := |W_{\Lambda_n}(C_{\pi/2}) + Z_n(\pi/2)|$. It suffices to verify that there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} P(H_{jn} \geq a) = O(e^{-a}), \quad j = 1, 2, 3$$

as $a \rightarrow \infty$. Here we only check it in case of $j = 1$. For the other two cases, the proofs are similar.

Since for all $n \geq 1$

$$\{W_{\Lambda_n}(C_\theta), \theta \in [0, \pi/2]\} \stackrel{d}{=} \{W(\Lambda_n(C_\theta)), \theta \in [0, \pi/2]\},$$

with W a standard Wiener process, we have

$$\begin{aligned} P(H_{1n} \geq a) &\leq P(\sup_{\theta \in [0, \pi/4]} |W(\Lambda_n(C_\theta))| \geq a/2) + P(\sup_{\theta \in [0, \pi/4]} |Z_n(\theta)| \geq a/2) \\ &\leq 2P(|W(\Lambda_n(C_{\pi/4}))| \geq a/2) + P(\sup_{\theta \in [0, \pi/4]} |Z_n(\theta)| \geq a/2). \end{aligned}$$

Clearly $\Lambda_n(C_{\pi/4}) \leq 1$ for all $n \geq 1$, and hence $\sup_{n \geq 1} P(|W(\Lambda_n(C_{\pi/4}))| \geq a/2) = O(e^{-a})$, as $a \rightarrow \infty$.

From Einmahl *et al.* (2001), one has $\sup_{x > 0} \lambda(x, 1) < \infty$ and $\sup_{y > 0} \lambda(1, y) < \infty$. Then by (4.4) there exists a constant $\lambda_0 > 0$ such that $\sup_{0 < x \leq 1} \lambda_n(x, 1) < \lambda_0$ and $\sup_{0 < y \leq 1} \lambda_n(1, y) <$

λ_0 for large n . Using (4.2) and the fact that Λ_n is a step function, one can prove with some effort that $\int_1^\infty \lambda_n(x, 1)dx \leq 2$ and $\int_1^\infty \lambda_n(1, y)dy \leq 2$ for sufficiently large n , hence by the definition of $Z_n(\theta)$, one has

$$\begin{aligned} & \sup_{\theta \in [0, \pi/4]} |Z_n(\theta)| \\ & \leq \lambda_0 \left| \int_0^1 \frac{W_{1n}(x)}{x} dx \right| + \lambda_0 \sup_{\theta \in [0, \pi/4]} \left| \tan \theta \int_1^{1/\tan \theta} \frac{W_{1n}(x)}{x} dx \right| + \lambda_0 \left| \int_0^1 \frac{W_{2n}(x)}{x} dx \right| + 2 |W_{2n}(1)| \\ & \leq \lambda_0 \left| \int_0^1 \frac{W_{1n}(x)}{x} dx \right| + \lambda_0 \sup_{x \geq 1} \left| \frac{W_{1n}(x)}{x} \right| + \lambda_0 \left| \int_0^1 \frac{W_{2n}(x)}{x} dx \right| + 2 |W_{2n}(1)|, \end{aligned}$$

for sufficiently large n . By Lemma 4.2(i), $\int_0^1 \frac{W_{1n}(x)}{x} dx$ and $\int_0^1 \frac{W_{2n}(x)}{x} dx$ have centered normal distributions with uniformly bounded variances for all $n \geq 1$. By Lemma 4.2(ii) there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{n \geq n_0} P(\lambda_0 \sup_{x \geq 1} |W_{1n}(x)|/x \geq a/8) \leq 2P(W(2) \geq a/(8\lambda_0)) = O(e^{-a})$$

as $a \rightarrow \infty$. Hence

$$\sup_{n \geq n_0} P(\sup_{\theta \in [0, \pi/4]} |Z_n(\theta)| \geq a/2) = O(e^{-a})$$

as $a \rightarrow \infty$. So $\sup_{n \geq n_0} P(H_{1n} \geq a) = O(e^{-a})$ as $a \rightarrow \infty$. \square

Lemma 4.4. *Under the conditions of Proposition 4.1, for each $0 \leq \eta < 1/2$*

$$\left\{ \frac{B_n(x, y)}{(x \vee y)^\eta}, (x, y) \in [0, 1]^2 \right\}_{n \geq 1}$$

is relatively compact.

Proof. By the definition of R_{1n} and R_{2n} , one has

$$\begin{aligned} R_{1n}(x, y) &= \frac{1}{2} k^{1/5} \Lambda_n([x - k^{-1/5}, x + k^{-1/5}] \times [0, \infty]) \\ &= \frac{1}{2} k^{1/5} \left(\frac{[k(x + k^{-1/5})]}{k} - \frac{[k(x - k^{-1/5})]}{k} \right) \\ &\leq 1 + 1/k^{4/5} \leq 2 \quad \text{if } k \geq 1. \end{aligned}$$

Also $R_{2n}(x, y) \leq 2$ for $k \geq 1$. Hence it is sufficient to prove

$$\{W_{R_n}(x, y)/(x \vee y)^\eta, x, y \in [0, 1]\}_{n \geq 1}, \quad \{W_{1n}(x)/x^\eta, x \in [0, 1]\}_{n \geq 1}, \quad \{W_{2n}(y)/y^\eta, y \in [0, 1]\}_{n \geq 1}$$

are relatively compact. Here we only show the proof of the first one. The proofs of the others are similar.

Setting $0/0 = 0$, by Lemma 4.1 it suffices to prove that for each positive ε , there exist a δ ($0 < \delta < 1$) and $n_0 \in \mathbb{N}$ (n_0 may depend on δ) such that

$$(4.1) \quad P \left(\sup_{\substack{x,y,u,v \in [0,1] \\ |x-u| \leq \delta, |y-v| \leq \delta}} \left| \frac{W_{\Lambda_n}([0,x] \times [0,y])}{(x \vee y)^\eta} - \frac{W_{\Lambda_n}([0,u] \times [0,v])}{(u \vee v)^\eta} \right| > \varepsilon \right) \leq \varepsilon, \quad n \geq n_0.$$

We partition the square $[0, 1] \times [0, 1]$ into m^2 ($m \in \mathbb{N}$) small squares, say $[0, 1] \times [0, 1] = \bigcup_{i=1}^m \bigcup_{j=1}^m \Delta_{ij}$, with $\Delta_{ij} := \{(x, y) : i\delta \leq x \leq (i+1)\delta, j\delta \leq y \leq (j+1)\delta\}$, $\delta := 1/m$ and $i, j = 0, 1, \dots, m-1$. In order to prove (4.1), it suffices to prove that for each positive ε , there exist a δ ($0 < \delta < 1$) and $n_0 = n_0(\delta) \in \mathbb{N}$ such that

$$(4.2) \quad \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P \left(\sup_{\Delta_{ij}} \left| \frac{W_{\Lambda_n}([0,x] \times [0,y])}{(x \vee y)^\eta} - \frac{W_{\Lambda_n}([0,i\delta] \times [0,j\delta])}{\delta^\eta(i \vee j)^\eta} \right| > \varepsilon \right) \leq \varepsilon, \quad n \geq n_0.$$

We consider the case $i \vee j \geq 1$ and the case $i = j = 0$ separately. Let's first look at the case $i \vee j \geq 1$. Assume $i > j$. Let $S(x, y) := [0, x] \times [0, y]$. Note that for $(x, y) \in \Delta_{ij}$

$$\begin{aligned} & \left| \frac{W_{\Lambda_n}([0,x] \times [0,y])}{(x \vee y)^\eta} - \frac{W_{\Lambda_n}([0,i\delta] \times [0,j\delta])}{\delta^\eta(i \vee j)^\eta} \right| \\ &= \left| \frac{W_{\Lambda_n}(S(x,y))}{x^\eta} - \frac{W_{\Lambda_n}(S(i\delta, j\delta))}{(i\delta)^\eta} \right| \\ &= \frac{|(i\delta)^\eta W_{\Lambda_n}(S(i\delta, j\delta)) + (i\delta)^\eta W_{\Lambda_n}(S(x,y) \setminus S(i\delta, j\delta)) - x^\eta W_{\Lambda_n}(S(i\delta, j\delta))|}{x^\eta (i\delta)^\eta} \\ &\leq \frac{|(i\delta)^\eta W_{\Lambda_n}(S(x,y) \setminus S(i\delta, j\delta)) - (x^\eta - (i\delta)^\eta) W_{\Lambda_n}(S(i\delta, j\delta))|}{(i\delta)^{2\eta}} \end{aligned}$$

(since $x \geq i\delta \geq y$). Hence

$$\begin{aligned} & P \left(\sup_{\Delta_{ij}} \left| \frac{W_{\Lambda_n}([0,x] \times [0,y])}{(x \vee y)^\eta} - \frac{W_{\Lambda_n}([0,i\delta] \times [0,j\delta])}{\delta^\eta(i \vee j)^\eta} \right| > \varepsilon \right) \\ &\leq P \left(\sup_{\Delta_{ij}} \left| \frac{W_{\Lambda_n}(S(x,y) \setminus S(i\delta, j\delta))}{(i\delta)^\eta} \right| > \frac{\varepsilon}{2} \right) + P \left(\sup_{\Delta_{ij}} \left| \frac{x^\eta - (i\delta)^\eta}{(i\delta)^{2\eta}} W_{\Lambda_n}(S(i\delta, j\delta)) \right| > \frac{\varepsilon}{2} \right) \\ &\leq 4P \left(\left| \frac{W_{\Lambda_n}(S((i+1)\delta, (j+1)\delta) \setminus S(i\delta, j\delta))}{(i\delta)^\eta} \right| > \frac{\varepsilon}{4} \right) + P \left(\left| \frac{(1+1/i)^\eta - 1}{(i\delta)^\eta} W_{\Lambda_n}(S(i\delta, j\delta)) \right| > \frac{\varepsilon}{2} \right). \end{aligned}$$

Since $\Lambda_n(S((i+1)\delta, (j+1)\delta) \setminus S(i\delta, j\delta)) \leq 2\delta + 4/k$ for all $i \vee j \geq 1$, there exists $n_* = n_*(\delta) \in \mathbb{N}$ such that $k_* = k(n_*) \geq 1/\delta$ and hence

$$\Lambda_n(S((i+1)\delta, (j+1)\delta) \setminus S(i\delta, j\delta)) \leq 6\delta, \quad n \geq n_*.$$

uniformly in $i \vee j \geq 1$. It follows that $(i\delta)^{-\eta} W_{\Lambda_n}(S((i+1)\delta, (j+1)\delta) \setminus S(i\delta, j\delta))$ has a normal distribution with mean zero and variance $\sigma_n^2(i, j)$ satisfying $\sigma_n^2(i, j) \leq 6\delta^{1-2\eta}$ for all

$i > j, i \geq 1$, and $n \geq n_*$. Hence for all $\varepsilon > 0$

$$\sup_{n \geq n_*} \sup_{i > j, i \geq 1} P(|(i\delta)^{-\eta} W_{\Lambda_n}(S((i+1)\delta, (j+1)\delta) \setminus S(i\delta, j\delta))| > \varepsilon/4) = O(e^{-\delta^{\eta-1/2}})$$

as $\delta \rightarrow 0$. On the other hand, note that $\frac{(1+1/i)^\eta - 1}{(i\delta)^\eta} W_{\Lambda_n}(S(i\delta, j\delta))$ has a normal distribution with mean zero and variance $\tilde{\sigma}_n^2(i, j)$ satisfying $\tilde{\sigma}_n^2(i, j) \leq (i\delta)^{1-2\eta}((1+1/i)^\eta - 1)^2 \leq 4\delta^{1-2\eta}$.

So

$$\sup_{n \geq n_*} \sup_{i > j, i \geq 1} P\left(\left|\frac{(1+1/i)^\eta - 1}{(i\delta)^\eta} W_{\Lambda_n}(S(i\delta, j\delta))\right| > \varepsilon/2\right) = O(e^{-\delta^{\eta-1/2}})$$

as $\delta \rightarrow 0$.

In case of $j > i, j \geq 1$ and case of $i = j \geq 1$, we can get similar results as above. Hence

$$(4.3) \quad \sup_{n \geq n_*} \sum_{i \vee j \geq 1}^{m-1} P\left(\sup_{\Delta_{ij}} \left| \frac{W_{\Lambda_n}([0, x] \times [0, y])}{(x \vee y)^\eta} - \frac{W_{\Lambda_n}([0, i\delta] \times [0, j\delta])}{\delta^\eta (i \vee j)^\eta} \right| > \varepsilon \right) = O(\delta^{-2} e^{-\delta^{\eta-1/2}})$$

as $\delta \rightarrow 0$.

Now let us look at the case $i = j = 0$. By Lemma 3.2 (in fact we can replace R by Λ_n in that lemma), one has

$$(4.4) \quad \sup_{n \geq 1} P\left(\sup_{x \vee y \leq \delta} \left| \frac{W_{\Lambda_n}([0, x] \times [0, y])}{(x \vee y)^\eta} \right| > \varepsilon \right) = O(e^{-\delta^{\eta-1/2}})$$

as $\delta \rightarrow 0$.

Since (4.3) and (4.4) imply (4.2), the result follows. \square

Lemma 4.5. *Under the conditions of Proposition 4.1, for each $0 \leq \eta < 1$*

$$\left\{ \frac{A_n(x, y)}{(x \vee y)^\eta}, (x, y) \in [0, 1]^2 \right\}_{n \geq 1}$$

is relatively compact.

Proof. The proof is similar to that of Lemma 4.4. We use the same notation for Δ_{ij} and S . We only need to check that for each positive ε , there exist a δ ($0 < \delta < 1$) and $n_0 \in \mathbb{N}$ such that

$$(4.5) \quad \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P\left(\sup_{\Delta_{ij}} \left| \frac{A_n(x, y)}{(x \vee y)^\eta} - \frac{A_n(i\delta, j\delta)}{\delta^\eta (i \vee j)^\eta} \right| > \varepsilon \right) \leq \varepsilon, \quad n \geq n_0.$$

We consider the case $i \vee j \geq 1$ and the case $i = j = 0$ separately. Let us first look at the case $i \vee j \geq 1$. In case of $i > j$, $i \geq 1$, note that for $(x, y) \in \Delta_{ij}$

$$\begin{aligned}
& |A_n(x, y)/(x \vee y)^\eta - A_n(i\delta, j\delta)/((i\delta) \vee (j\delta))^\eta| \\
&= \left| (x^{1-\eta} - (i\delta)^{1-\eta})(W_{\Lambda_n}(C_{\pi/2}) + Z_n(\pi/2)) - (x^{1-\eta} - (i\delta)^{1-\eta}) \int_{\arctan y/x}^{\pi/4} \frac{1}{\cos^2 \theta} (W_{\Lambda_n}(C_\theta) + Z_n(\theta)) d\theta \right. \\
&\quad \left. + (i\delta)^{1-\eta} \int_{\arctan j/i}^{\arctan y/x} \frac{1}{\cos^2 \theta} (W_{\Lambda_n}(C_\theta) + Z_n(\theta)) d\theta \right| \\
&\leq (x^{1-\eta} - (i\delta)^{1-\eta}) \left(1 + \int_0^{\pi/4} (\cos \theta)^{-2} d\theta \right) H_n + (i\delta)^{1-\eta} H_n \int_{\arctan j/i}^{\arctan(j+1)/i} (\cos \theta)^{-2} d\theta \\
&\leq (i\delta)^{1-\eta} ((1 + 1/i)^{1-\eta} - 1)(1 + \pi/2) H_n + (i\delta)^{1-\eta} \left(\arctan \frac{j+1}{i} - \arctan \frac{j}{i} \right) 2 H_n
\end{aligned}$$

where H_n is defined in Lemma 4.3. Since $(i\delta)^{1-\eta}((1+1/i)^{1-\eta}-1) \leq \delta^{1-\eta}$ and $(i\delta)^{1-\eta}(\arctan \frac{j+1}{i} - \arctan \frac{j}{i}) = O(\delta^{1-\eta})$ as $\delta \rightarrow 0$ and uniformly in i, j ($i > j, i \geq 1$), then by Lemma 4.3 there exists $n_0 \in \mathbb{N}$ such that for some $M > 0$

$$\begin{aligned}
(4.6) \quad & \sup_{n \geq n_0} \sup_{i > j, i \geq 1} P(\sup_{\Delta_{ij}} |A_n(x, y)/(x \vee y)^\eta - A_n(i\delta, j\delta)/((i\delta) \vee (j\delta))^\eta| > \varepsilon/2) \\
& \leq \sup_{n \geq n_0} P(M\delta^{1-\eta} H_n > \varepsilon/2) \leq \sup_{n \geq n_0} P(H_n \geq \delta^{(\eta-1)/2}) = O(e^{-\delta^{(\eta-1)/2}})
\end{aligned}$$

as $\delta \rightarrow 0$.

In case of $j > i, j \geq 1$ and case of $i = j \geq 1$ we can get a similar result as (4.6). Hence

$$(4.7) \quad \sup_{n \geq n_0} \sum_{i \vee j \geq 1}^m P(\sup_{\Delta_{ij}} |A_n(x, y)/(x \vee y)^\eta - A_n(i\delta, j\delta)/((i\delta) \vee (j\delta))^\eta| > \varepsilon) = O(\delta^{-2} e^{-\delta^{(\eta-1)/2}})$$

as $\delta \rightarrow 0$.

Now let's consider the case $i = j = 0$ and w.l.o.g. assume $y \geq x$. Then for $0 \leq x \leq y \leq \delta$

$$\begin{aligned}
& |A_n(x, y)/(x \vee y)^\eta| \\
&= \left| xy^{-\eta} W_{\Lambda_n}(C_{\pi/2}) + Z_n(\pi/2) + y^{1-\eta} \int_{\pi/4}^{\arctan y/x} (\sin \theta)^{-2} (W_{\Lambda_n}(C_\theta) + Z(\theta)) d\theta \right| \\
&\leq y^{1-\eta} \left(xy^{-1} + \int_{\pi/4}^{\pi/2} (\sin \theta)^{-2} d\theta \right) H_n \leq \delta^{1-\eta} (1 + \pi/2) H_n.
\end{aligned}$$

Hence

$$(4.8) \quad \sup_{n \geq n_0} P(\sup_{x \vee y \leq \delta} |A_n(x, y)/(x \vee y)^\eta| > \varepsilon) = O(e^{-\delta^{(\eta-1)/2}})$$

as $\delta \rightarrow 0$.

Now (4.7) and (4.8) imply (4.5). □

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