Insurance: Mathematics and Economics 47 (2010) 187–189

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A note on additive risk measures in rank-dependent utility

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ARTICLE INFO

Article history:
Received February 2010
Received in revised form
May 2010
Accepted 12 May 2010

JEL classification:
D81
G20

MSC:
62P05

Keywords:
Decision-making
Measure of risk
Premium principle
Equivalent utility
Rank-dependent utility
Exponential utility
Axiomatization
Additivity

ABSTRACT

This note proves that risk measures obtained by applying the equivalent utility principle in rank-dependent utility are additive if and only if the utility function is linear or exponential and the probability weighting (distortion) function is the identity.

1. Introduction

In this note, we consider rank-dependent utility of a risk. Rank-dependent utility was first proposed by Quiggin (1982) under the guise of anticipated utility; see also Yaari (1987). It coincides with Choquet expected utility of Schmeidler (1989) for *decision-making under risk*, that is, decision-making in settings where the probability measure is objective, and known and given in advance, which is the setting considered here. It further coincides with (and was in fact at the basis of) cumulative prospect theory of Tversky and Kahneman (1992) for decision-making under risk, when restricting to either positive or negative outcomes. Kahneman was awarded the 2002 Nobel Memorial Prize in Economics for his work in prospect theory.

From this rank-dependent utility, we derive a risk measure $\rho[X]$ as that displacement of the risk X that makes the rank-dependent utility $\pi[X - \rho[X]]$ of the shifted risk equal to zero; see Denuit et al. (2006) and Goovaerts et al. (forthcoming). Throughout, realizations of X designate monetary payoffs.

We show that if the rank-dependent utility $\pi[X]$ involves a concave utility function next to its probability weighting (or distortion) function, and moreover the risk measure $\rho[X]$ is *additive* for independent risks, then the probability weighting function must be linear and the utility function must be linear or exponential. Thus, we generalize the following results. Gerber (1974a, 1985) proves that in the absence of a distortion function (that is, if it is the identity), the utility function must be linear or exponential. Heilpern (2003) proves that, with linear or exponential utility, the distortion function must be linear to make ρ additive. Our proof proceeds by using the additivity property for simple Bernoulli risks to derive a differential equation for the utility function u, and proving that its Maclaurin expansion has the required coefficients.

Additivity of risk measures used for premium calculation was advocated already by Borch (1962, p. 429): "It is natural to require that the company shall receive the same amount whether it accepts the two [independent] portfolios separately or in one single transaction".

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doi:10.1016/j.insmatheco.2010.05.003

2. The result

Our main result is the following.

Theorem 2.1. *Consider a utility function u and a distortion function g, where*

(i) $u : \mathbb{R} \to \mathbb{R}$ *is a function with a Maclaurin expansion, normalized in the sense that* $u(0) = 0, u'(0) = 1$*, strictly increasing and concave, so* $u' > 0, u'' \le 0$*;*
(ii) $g : [0, 1] \to [0, 1]$ *has* $g(0) = 0, g(1) = 1, g' \ge 0, g''$ *exists and* $g'(1) \ne 0$.

For any random variable X, consider its rank-dependent utility

$$\pi[X] := \int_{-\infty}^{0} (g(\mathbb{P}[X > x]) - 1)\,\mathrm{d}u(x) + \int_{0}^{+\infty} g(\mathbb{P}[X > x])\,\mathrm{d}u(x).$$

Now define the risk measure ρ *as the solution to the equation* $\pi[X - \rho] = 0$*, that is, by the principle of equivalent utility. This risk measure is* additive *if and only if g is the identity function and u is exponential or linear.*

Proof. Notice first that the assumptions that u has a Maclaurin expansion and is strictly increasing with $u'(0) = 1$, together with $g(0) = 0, g(1) = 1$ and $g' \ge 0$, guarantee that the equation $\pi[X - \rho] = 0$ admits at most one solution. If, for given X, u, g, the equation $\pi[X - \rho] = 0$ does not have a solution, X is uninsurable under rank-dependent utility with utility function u and distortion function g.

The 'if' part of the theorem is trivial. Let us prove the 'only if' part. Let $X_{c,t}$ be a Bernoulli risk with parameters c, t such that $\mathbb{P}[X_{c,t} = 0] = t$ and $\mathbb{P}[X_{c,t} = c] = 1 - t$, so

$$\mathbb{P}[X_{c,t} - \rho > x] = \mathbb{P}[X_{c,t} > x + \rho] = \begin{cases} 1 & \text{for } x < -\rho, \\ 1 - t & \text{for } -\rho \le x < c - \rho, \\ 0 & \text{for } c - \rho \le x. \end{cases}$$

For such a risk, we get, using $0 \le \rho \le c$,

$$\begin{aligned}\pi[X_{c,t} - \rho] &= -\int_{-\rho}^{0} (1 - g(1-t))\,\mathrm{d}u(x) + \int_{0}^{c-\rho} g(1-t)\,\mathrm{d}u(x) \\ &= (1 - g(1-t))u(-\rho) + g(1-t)u(c-\rho) \\ &= 0.\end{aligned}$$

Now we consider $\rho = \rho(t)$ as a function of t while keeping c fixed; we write $\rho'(t)$ and $\rho''(t)$ for its derivatives with respect to t. Taking the first derivative with respect to t in the equation above, we find

$$g'(1-t)u(-\rho) + (1 - g(1-t))u'(-\rho)\left(-\rho'(t)\right) - g'(1-t)u(c-\rho) + g(1-t)u'(c-\rho)\left(-\rho'(t)\right) = 0.$$

Now we set $t = 0$, so $\rho = c$, and the first term becomes $g'(1)u(-c)$. By our assumptions (i) and (ii), the second and third terms vanish, and the fourth term reduces to $-\rho'(0)$. So we have

$$g'(1)u(-c) = \rho'(0). \tag{1}$$

We have excluded $g'(1) = 0$ in (ii). Note that if this should hold, so $\rho'(0) = 0$, the risk measure ρ would be insensitive to losing an amount c with probability 1.

Differentiating once more with respect to t yields

$$\begin{aligned}&-g''(1-t)u(-\rho) - 2g'(1-t)u'(-\rho)\rho'(t) \\ &\quad + (1 - g(1-t))\left(u''(-\rho)\left(\rho'(t)\right)^2 - u'(-\rho)\rho''(t)\right) \\ &\quad + g''(1-t)u(c-\rho) + 2g'(1-t)u'(c-\rho)\rho'(t) \\ &\quad + g(1-t)\left(u''(c-\rho)\left(\rho'(t)\right)^2 - u'(c-\rho)\rho''(t),\right) = 0,\end{aligned}$$

which, by our assumptions, for $t = 0$ reduces to

$$-g''(1)u(-c) - 2g'(1)u'(-c)\rho'(0) + 2g'(1)\rho'(0) + u''(0)\left(\rho'(0)\right)^2 = \rho''(0). \tag{2}$$

Now let $S := X_{c,t} + Y_{c,t}$, where $X_{c,t}$ and $Y_{c,t}$ are two independent and identically distributed Bernoulli risks, so

$$\mathbb{P}[S - 2\rho > x] = \mathbb{P}[S > x + 2\rho] = \begin{cases} 1 & \text{for } x < -2\rho; \\ 1 - t^2 & \text{for } -2\rho \le x < c - 2\rho; \\ (1-t)^2 & \text{for } c - 2\rho \le x < 2(c-\rho); \\ 0 & \text{for } 2(c-\rho) \le x. \end{cases}$$

Then, provided that $c \le 2\rho \le 2c$,

$$\begin{aligned}\pi[S - 2\rho] &= -\int_{-2\rho}^{c-2\rho} (1 - g(1-t^2))\,\mathrm{d}u(x) - \int_{c-2\rho}^{0} (1 - g((1-t)^2))\,\mathrm{d}u(x) + \int_{0}^{2(c-\rho)} g((1-t)^2)\,\mathrm{d}u(x) \\ &= (1 - g(1-t^2))u(-2\rho) + (g(1-t^2) - g((1-t)^2))u(c-2\rho) + g((1-t)^2)u(2(c-\rho)) \\ &= 0.\end{aligned}$$

The first derivative with respect to t at $t = 0$ (where $c \le 2\rho \le 2c$ is satisfied), by the assumed additivity of ρ, again leads to Eq. (1):

$$g'(1)u(-c) = \rho'(0).$$

The second derivative at $t = 0$, again assuming additivity of ρ, yields

$$2g'(1)u(-2c) - 4(g'(1) + g''(1))u(-c) - 8g'(1)u'(-c)\rho'(0) + 8g'(1)\rho'(0) + 4u''(0)(\rho'(0))^2 - 2\rho''(0) = 0.$$

Substituting (1) and (2) in the above expression, we obtain

$$\begin{aligned}&2g'(1)u(-2c) - 4g'(1)u(-c) - 2g''(1)u(-c) \\ &\quad - 4(g'(1))^2u'(-c)u(-c) + 4(g'(1))^2u(-c) \\ &\quad + 2(g'(1))^2u''(0)(u(-c))^2 = 0,\end{aligned}$$

or, equivalently,

$$u(-c)\left(g''(1) - 2(g'(1))^2 + 2g'(1)\right) - u(-2c)g'(1) - (u(-c))^2u''(0)(g'(1))^2 + u'(-c)u(-c)2(g'(1))^2 = 0,$$

or, with $g'(1) \ne 0$,

$$u(-c)\left(\frac{g''(1)}{(g'(1))^2} - 2 + 2\frac{1}{g'(1)}\right) - u(-2c)\frac{1}{g'(1)} - (u(-c))^2u''(0) + 2u'(-c)u(-c) = 0. \tag{3}$$

Recall that $u(0) = 0$ and $u'(0) = 1$. To simplify notation, write

$$d := -c, \qquad \delta := \frac{1}{g'(1)}, \qquad \varepsilon := \frac{g''(1)}{(g'(1))^2} - 2 + 2\frac{1}{g'(1)}.$$

The Maclaurin expansion for $u(d)$ and its derivative can be written as

$$u(d) = d + \sum_{j=2}^{\infty} \frac{u^{(j)}(0)}{j!}d^j \quad \text{and} \quad u'(d) = 1 + \sum_{j=2}^{\infty} \frac{u^{(j)}(0)}{(j-1)!}d^{j-1}.$$

Eq. (3) obtained above can be rewritten as

$$\varepsilon u(d) - \delta u(2d) - (u(d))^2\,u''(0) + 2u'(d)u(d) = 0, \quad \forall d. \tag{4}$$

All coefficients of d^j should be equal to zero on the left-hand side. There is no constant, and the coefficient of d^1 leads to

$$\varepsilon - 2\delta - 0 + 2 = \frac{g''(1)}{(g'(1))^2} = 0.$$

Therefore $g''(1) = 0$.

Equating the coefficient of d^2 to zero gives that we must have

$$u''(0)(1-\delta) = 0.$$

Evidently, this holds if $\delta = 1$, which means that $g'(1) = 1$. But when $g'(1) = 1$ as well as $g''(1) = 0$, the differential equation (3) coincides with the one in Gerber (1974a, 1985), for which he proves that it is only satisfied if u is exponential or linear. Then, by Heilpern (2003), g must be linear; hence, g is the identity function.

For the case $\delta = 1/g'(1) \neq 1$, we will show that u must be linear, and then the result of the theorem follows again directly from Heilpern (2003). By the last equation, from $\delta \neq 1$, it follows that $u''(0) = 0$ must hold. To show that then $u^{(j)}(0) = 0$ also for all $j = 3, 4, \ldots$, we use induction. So assume that $u^{(j)}(0) = 0$ holds for all $j = 2, \ldots, n-1$. Then the coefficient of d^n in the Maclaurin expansion of (4) can be written as

$$2(\delta-1)\frac{u^{(n)}(0)}{n!} - \delta 2^n \frac{u^{(n)}(0)}{n!} + 0 + 2\left(\frac{u^{(n)}(0)}{n!} + 0\right)$$
$$= \frac{u^{(n)}(0)}{n!}\left(2\delta - 2^n\delta\right).$$

Since $\delta > 0$ must hold, (4) holding implies that $u^{(n)}(0) = 0$, eventually for all $n = 2, 3, \ldots$. By Heilpern (2003), g must then be linear, which contradicts the stated assumption that $\delta = 1/g'(1) \neq 1$. This means that $\delta = 1/g'(1) \neq 1$ cannot hold. This completes the proof. □

Remark 2.1. We required that the function $u(\cdot)$ has a Maclaurin expansion, to be able to identify it from its derivatives at 0 only. This is not a trivial assumption; for example, defining $f(x) = \exp(-1/x^2)$ with $f(0) = 0$, it is easy to see that u and $u + f$ have the same derivatives at 0.

3. Concluding remarks

Goovaerts et al. (2004) prove a general representation result for additive risk measures. Their result entails that risk measures are normalized and additive, and respect exponential order (hence, are monotone) if and only if they are mixtures of exponential premiums; see also Gerber and Goovaerts (1981) and Goovaerts and Laeven (2008) for related results.

In contrast to Goovaerts et al. (2004), the additive ρ solving $\pi[X-\rho] = 0$ we find does not permit a (non-degenerate) mixture function in its representation. This can be explained by either one of the following two reasons. (i) The mixture function is not compatible with the comonotone independence axiom needed to axiomatize rank-dependent utility. (ii) The mixture function is not compatible with the iterativity property obtained here.

With respect to (ii) we note the following: Gerber (1974b) proves that a risk measure that satisfies a certain continuity condition is iterative if and only if it is a mean value principle. Furthermore, as is well known, the mean value principle is additive (hence, translation invariant) if and only if it is an exponential premium. The mixture of exponential premiums of Goovaerts et al. (2004) satisfies the particular continuity condition of Gerber (1974b), and furthermore is additive. Hence, we conclude that it is iterative if and only if the mixture function is degenerate.

A second remark we want to make is that one may wonder whether a result similar to our main theorem holds for Choquet expected utility of Schmeidler (1989) or maxmin expected utility of Gilboa and Schmeidler (1989). But requiring additivity, or rather $\mathbb{P}$-additivity, in settings of *decision-making under uncertainty*, where the probability measure need not be known or given in advance, seems quite unnatural. For decision-making under risk, Choquet expected utility coincides with rank-dependent utility and maxmin expected utility with Von Neumann and Morgenstern (1944) expected utility. This means that for decision-making under risk, the main result contained in this note holds across the dominant decision-making paradigms.

Acknowledgements

We are grateful to Hans Gerber and to two anonymous referees for their valuable comments and suggestions. Marc Goovaerts acknowledges the financial support of the GOA Grant 2007 ("Risk Modeling and Valuation of Insurance and Financial Cash Flows, with Applications to Pricing, Provisioning and Solvency"). Roger Laeven acknowledges the financial support of the Netherlands Organization for Scientific Research (NWO Grant No. 42511013, NWO VENI Grant 2006 and NWO VIDI Grant 2009).

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