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Worst VaR scenarios with given marginals and measures of association

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ABSTRACT

This paper studies the problem of finding best-possible upper bounds on the Value-at-Risk for a function of two random variables when the marginal distributions are known and additional nonparametric information on the dependence structure, such as the value of a measure of association, is available. The same problem for the Tail-Value-at-Risk is also briefly discussed.

1. Introduction

This paper studies generalized versions of the following problem, which was attributed to A.N. Kolmogorov by Makarov (1981): Let X and Y be two random variables with given distribution functions F_1 and F_2, respectively. Let G_ψ denote the distribution function of a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ (e.g., the sum) of X and Y. Find $\underline{G}_\psi(x) = \inf\{G_\psi(x)\}$, where the infimum is taken over the *Fréchet–Hoeffding class* $\mathbf{F}(F_1, F_2)$ consisting of all joint distribution functions with marginals F_1 and F_2. In this paper, we will be concerned with generalized versions allowing for partial information on the dependence structure between X and Y, in which case the infimum is taken over well-defined subclasses of $\mathbf{F}(F_1, F_2)$.

Besides being theoretically challenging, the problem described above is highly relevant in risk management: it is equivalent to finding worst case scenarios for the Value-at-Risk for a function of two risks (random variables) when the marginal distributions of the risks are known but the dependence structure between the risks is unknown, or, as in this paper, only partially known.

In the complete absence of partial information on the dependence structure, the problem of Kolmogorov was solved by Makarov (1981) for the case that G is the distribution function of the sum of X and Y with given marginals F_1 and F_2. Using different routes, the same problem was solved by Rüschendorf (1982, dual approach) and by Frank et al. (1987, copula-based approach).

The presence of partial information on the dependence structure may conceivably allow obtaining tighter bounds. For instance, if a lower bound on the copula of X and Y – sharper than the Fréchet–Hoeffding lower bound – is available, Williamson and Downs (1990) provide a general method to produce tighter bounds; see Section 2 for details. Such a lower bound need not be described parametrically or even have a closed-form expression. It could, for example, be represented in a lookup table. It motivates the search for lower bounds on the copula. However, in many

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Fig. 1. The Fréchet–Hoeffding bounds and the product copula.

cases, partial information on the dependence structure does not trivially translate into a lower bound on the copula. In this paper we will carry out some of such non-trivial translations, establishing some new copula bounds in the presence of partial nonparametric dependence information.

One of our aims is to investigate and illustrate the effectiveness (or in some cases rather, lack thereof) of different types of information on the dependence structure in bounding the Value-at-Risk. To some extent the paper is meant as an educative warning that, contrary to what is quite often thought in the insurance and financial industry, the Value-at-Risk may vary widely even when the marginals and a nonparametric dependence measure, such as the value of a measure of association, are fixed. Readers should thus be careful of adopting in this context the commonly used multivariate inference techniques that *implicitly* assume otherwise. The same problem for the Tail-Value-at-Risk is also briefly discussed.

Other work related to the problem of Kolmogorov includes, without being exhaustive, Denuit et al. (1999), Nelsen et al. (2001, 2004), Dhaene et al. (2002), Nelsen and Úbeda Flores (2004), Denuit et al. (2005), Embrechts et al. (2005), Embrechts and Puccetti (2006) and Laeven (2009).

The outline of this paper is as follows: In Section 2 we briefly review some probabilistic theory of copulas in view of the problem under study. In Sections 3–5 we derive new copula bounds in the presence of nonparametric information on the dependence structure. In Section 6 we use these copula bounds to obtain bounds on the Value-at-Risk and study the effectiveness of different types of nonparametric dependence information in bounding the Value-at-Risk. Section 7 briefly studies worst Tail-Value-at-Risk scenarios. We finish with an open problem in Section 8 and some concluding remarks in Section 9.

2. Preliminaries

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider two continuous random variables (r.v.'s) X and Y defined on it. We denote by $F(x, y) = \mathbb{P}[X \leq x, Y \leq y]$ the joint distribution function (d.f.) of (X, Y) and denote by $F_1(x) = \mathbb{P}[X \leq x]$ and $F_2(y) = \mathbb{P}[Y \leq y]$ the marginal d.f.'s. We assume throughout that the marginal d.f.'s are given while the joint d.f. is unknown.

A bivariate copula is a function $C : [0, 1]^2 \to [0, 1]$ that satisfies the boundary conditions

$$C(u, 0) = C(0, v) = 0,$$
$$C(u, 1) = u, \qquad C(1, v) = v,$$

for every $u, v \in [0, 1]$, and is 2-increasing on $[0, 1]^2$, i.e.,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \tag{1}$$

for all $u_1, u_2, v_1, v_2 \in [0, 1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$. Equivalently, a bivariate copula is a bivariate d.f. with domain $[0, 1]^2$ and uniform $(0, 1)$ marginals. Every bivariate copula C satisfies the *Fréchet–Hoeffding inequality*

$$\begin{aligned} W(u, v) &= \max\{0;\ u + v - 1\} \\ &\leq C(u, v) \leq \min\{u; v\} = M(u, v), \end{aligned}$$

for every $(u, v) \in [0, 1]^2$. Here, the Fréchet–Hoeffding bounds W and M are themselves bivariate copulas. A third copula that plays an important role is the *product copula* $\Pi(u, v) = uv$. Fig. 1 plots the copulas W, M and Π.

It is straightforward to verify that for a given bivariate copula C and given marginals F_1 and F_2, the function F defined by

$$F(x, y) = C(F_1(x), F_2(y)), \quad (x, y) \in \mathbb{R}^2, \tag{2}$$

is a bivariate d.f. with marginals F_1 and F_2. Sklar (1959) proved that also the converse is true: for a given bivariate d.f. F with