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Original Article

The tail probability of discounted sums of Pareto-like losses in insurance

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In an insurance context, the discounted sum of losses within a finite or infinite time period can be described as a randomly weighted sum of a sequence of independent random variables. These independent random variables represent the amounts of losses in successive development years, while the weights represent the stochastic discount factors. In this paper, we investigate the problem of approximating the tail probability of this weighted sum in the case when the losses have Pareto-like distributions and the discount factors are mutually dependent. We also give some simulation results.

Keywords: Asymptotics; (Log)elliptical distribution; (Log)normal variance-mean mixed distribution; Pareto-like distribution; Tail probability.

Mathematics Subject Classification: 62P05; 62E20; 91B30.

1. Introduction

Motivated by the work of Resnick & Willekens [1], we investigate the tail probabilities of the randomly weighted sums

$$\sum_{k=1}^{n} \theta_k X_k, \quad n = 1, 2, \ldots, \tag{1.1}$$

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and their maxima. Here $\{X_n, n=1,2,\ldots\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with generic random variable X and common distribution function $F=1-\overline{F}$, while $\{\theta_n, n=1,2,\ldots\}$ is a sequence of dependent nonnegative random variables, independent of the sequence $\{X_n, n=1,2,\ldots\}$.

The randomly weighted sums (1.1) and their maxima are often encountered in actuarial and economic situations. See the following examples:

Example 1.1. Just as in Nyrhinen [2] and Tang & Tsitsiashvili [3,4], consider a discrete time risk model, in which the surplus of the insurance company is invested into a risky asset that generates a random, possibly negative, return rate in each year. Denote by $A_n \in (-\infty,\infty)$ the net income (the total premium income minus the total claim amount) within year n and by $R_n \in (-1,\infty)$ the random return rate in year n, $n=1,2,\ldots$. Let the initial surplus be $x \geq 0$. Hence, if we assume that the net income A_n is calculated at the end of year n, then the surplus, denoted by U_n, accumulated till the end of year n satisfies the recurrence equation

$$U_0 = x \geq 0, \quad U_n = (1+R_n)U_{n-1} + A_n, \quad n = 1, 2, \ldots. \tag{1.2}$$

We define the finite time ruin probability as

$$\psi(x;n) = \Pr\left(\min_{0\leq k\leq n} U_k < 0 \,\middle|\, U_0 = x\right) \tag{1.3}$$

and the infinite time ruin probability as

$$\psi(x) = \lim_{n\to\infty} \psi(x;n) = \Pr\left(\min_{0\leq k<\infty} U_k < 0 \,\middle|\, U_0 = x\right). \tag{1.4}$$

Now write

$$X_n = -A_n, \quad Y_n = \frac{1}{1+R_n}, \quad n = 1, 2, \ldots. \tag{1.5}$$

The random variable X_n is the net payout within year n and the random variable Y_n is the discount factor from year n to year $n-1$, $n=1,2,\ldots$. In the terminology of Norberg [5] and Tang & Tsitsiashvili [3,4], we call X_n, $n=1,2,\ldots$, the insurance risks and Y_n, $n=1,2,\ldots$, the financial risks.

The discounted value of the surplus process U_n, denoted by $\tilde{U}_n$, is defined by

$$\tilde{U}_0 = x, \quad \tilde{U}_n = \prod_{i=1}^{n} Y_i U_n, \qquad n = 1, 2, \ldots.$$

By repeatedly substituting (1.2) in the above expression, we find that $\tilde{U}_n$ can also be expressed as

$$\tilde{U}_0 = x, \quad \tilde{U}_n = x - \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i = x - W_n, \qquad n = 1, 2, \ldots.$$

One sees that the W_n introduced above (with $W_0=0$), which denotes the total discounted amount of losses by the end of year n, is of the form (1.1) with $\theta_k = \prod_{i=1}^{k} Y_i$, which is a product of positive random variables. We rewrite the ruin probabilities in terms of W_k, $k=0,1,2,\ldots$, as

$$\psi(x;n)=\Pr\left(\max_{0\le k\le n} W_k > x \,\middle|\, U_0=x\right)$$

and

$$\psi(x)=\Pr\left(\max_{0\le k<\infty} W_k > x \,\middle|\, U_0=x\right). \quad \square$$

Example 1.2. In Nyrhinen [2] and Tang & Tsitsiashvili [3,4], it was assumed that the net incomes A_n, $n=1,2,\ldots$, constitute a sequence of i.i.d. random variables, that the return rates R_n, $n=1,2,\ldots$, also constitute a sequence of i.i.d. random variables, and that the two sequences $\{A_n, n=1,2,\ldots\}$ and $\{R_n, n=1,2,\ldots\}$ are independent. A particular case is the well-known Black-Scholes-Merton model, in which the financial risks Y_n, $n=1,2,\ldots$, are assumed to be i.i.d. and lognormally distributed. $\square$

Example 1.3. Since the assumption of independent return rates made in Example 1.2 is generally considered unrealistic, it is desirable to incorporate some dependence structure in the financial risks. A natural extension is to assume that the log returns follow a multivariate normal distribution, or, more precisely, that there are sequences $\{\mu_n, n=1,2,\ldots\}$ and $\{\sigma_{ij}, i,j=1,2,\ldots\}$ such that for each n, the vector

$$(Z_1,Z_2,\ldots,Z_n)=(-\log Y_1,-\log Y_2,\ldots,-\log Y_n) \tag{1.6}$$

has a multivariate normal distribution with mean vector

$$\mu_n=(\mu_1,\mu_2,\ldots,\mu_n)$$

and covariance matrix

$$\Sigma_n=\begin{pmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}\end{pmatrix}.$$

Clearly, this assumption is very convenient in calculation because of the attractive properties of the multivariate normal distribution.

However, this assumption has also been questioned recently by Bingham *et al.* [6], because "the empirical evidence shows that most financial data exhibit both pronounced asymmetry and much heavier tail behaviour than is consistent with normality." Following the work of Bingham *et al.* [6], we shall circumvent the limitations of the Black-Scholes-Merton framework by assuming that vector (1.6) either has a multivariate normal variance-mean mixture with some mixing law, or follows a multivariate elliptical distribution. $\square$

Keeping these examples in mind, we shall investigate the tail probability of the randomly weighted sums (1.1) and their maxima, under the assumptions that the distribution function F is Pareto-like and that the random weights $\{\theta_n, n=1,2,\ldots\}$ satisfy some conditions.

The remaining part of the paper is organized as follows: Section 2 gives the main results and some remarks; Section 3 considers two special cases when the quantities involved in the asymptotic results can be handled; and Section 4 proves the main results, after recalling several known results.

2. Main results

Throughout this paper, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \lesssim b(x)$ if lim sup $a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if lim inf $a(x)/b(x) \geq 1$, and write $a(x) \sim b(x)$ if both.

Recall the randomly weighted sums (1.1). We assume that the right tail of F is regularly varying in the sense that there exist a constant $\alpha \geq 0$ and a slowly varying function $L(\cdot)$ such that

$$\overline{F}(x) = x^{-\alpha}L(x), \quad x > 0. \tag{2.1}$$

We designate the fact (2.1) by $F \in \mathcal{R}_{-\alpha}$. This class contains the famous Pareto distributions. By the well-known representation theorem for slowly varying functions (see Theorem 1.3.1 of Bingham *et al.* [7]), we have that for a distribution function $F \in \mathcal{R}_{-\alpha}$ and any $\beta > \alpha$,

$$x^{-\beta} = o(\overline{F}(x)). \tag{2.2}$$

More generally, the class $\mathcal{R}$ is the union of all $\mathcal{R}_{-\alpha}$ over the range $0 \leq \alpha < \infty$. For more details on the class $\mathcal{R}$, we refer the reader to Bingham *et al.* [7].

Now we state the main contributions of this paper. The first result deals with the case of randomly weighted sums of finite summands.

THEOREM 2.1. *Consider the randomly weighted sums (1.1) and let $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. We have*

$$\Pr\left(\max_{1 \leq m \leq n} \sum_{k=1}^{m} \theta_k X_k > x\right) \sim \Pr\left(\sum_{k=1}^{n} \theta_k X_k > x\right) \sim \overline{F}(x) \sum_{k=1}^{n} \mathrm{E}\theta_k^{\alpha} \tag{2.3}$$

if there exists some $\delta > 0$ such that

(1) $\mathrm{E}\theta_k^{\alpha+\delta} < \infty$ *for each* $1 \leq k \leq n$. □

By the result of Theorem 2.1, we have under the same assumptions that for all $n = 1, 2, \ldots$

$$\liminf_{x \to \infty} \frac{1}{\overline{F}(x)} \Pr\left(\max_{1 \leq m < \infty} \sum_{k=1}^{m} \theta_k X_k > x\right) \geq \sum_{k=1}^{n} \mathrm{E}\theta_k^{\alpha}. \tag{2.4}$$

Hence, by the arbitrariness of n it follows that

$$\liminf_{x\to\infty}\frac{1}{\overline{F}(x)}\Pr\left(\max_{1\le m<\infty}\sum_{k=1}^{m}\theta_kX_k>x\right)\ge\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha} \tag{2.5}$$

regardless of whether $\Sigma_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha}$ converges.

For any real number x, we write its positive part by $x^+=x_+=\max\{x,0\}$. The following result extends Theorem 2.1 to the case of infinite sums.

THEOREM 2.2. *For the randomly weighted sums (1.1) with $F\in\mathcal{R}_{-\alpha}$ for some $\alpha>0$, we have*

$$\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\theta_kX_k>x\right)\sim\Pr\left(\sum_{k=1}^{\infty}\theta_kX_k^+>x\right)\sim\overline{F}(x)\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha} \tag{2.6}$$

if one of the following assumptions holds:

(2) $0<\alpha<1$ *and*

$$\sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha+\delta}<\infty \quad and \quad \sum_{k=1}^{\infty}\mathrm{E}\theta_k^{\alpha-\delta}<\infty \quad for\ some\ \delta>0; \tag{2.7}$$

(3) $1\le\alpha\le\infty$ *and*

$$\sum_{k=1}^{\infty}(\mathrm{E}\theta_k^{\alpha+\delta})^{\frac{1}{\alpha+\delta}}<\infty \quad and \quad \sum_{k=1}^{\infty}(\mathrm{E}\theta_k^{\alpha-\delta})^{\frac{1}{\alpha+\delta}}<\infty \quad for\ some\ \delta>0. \tag{2.8}$$

□

REMARK 2.1. Both Theorems 2.1 and 2.2 do not require any information about the dependence structure of the sequence $\{\theta_n, n=1,2,\ldots\}$. □

REMARK 2.2. Recall Example 1.2, where the random variables θ_k in (1.1) are interpreted as discount factors from time k to time 0 and are expressed as

$$\theta_k=\prod_{j=1}^{k}Y_j, \quad k=1,2,\ldots, \tag{2.9}$$

with i.i.d. nonnegative random variables $\{Y_n, n=1,2,\ldots\}$. Clearly, in this standard case, assumption (1) of Theorem 2.1 is equivalent to

(4) $\mathrm{E}Y_1^{\alpha+\delta}<\infty$ for some $\delta>0$,

and assumptions (2) and (3) of Theorem 2.2 are equivalent to

(5) $\mathrm{E}Y_1^{\alpha\pm\delta}<1$ for some $\delta>0$.

Under these assumptions, it follows from Theorems 2.1 and 2.2 that

$$\psi(x;n)\sim\overline{F}(x)\frac{\mathrm{E}Y_1^{\alpha}(1-(\mathrm{E}Y_1^{\alpha})^n)}{1-\mathrm{E}Y_1^{\alpha}}$$

and that

$$\psi(x) \sim \overline{F}(x) \frac{\mathrm{E}\, Y_1^{\alpha}}{1 - \mathrm{E}\, Y_1^{\alpha}}.$$

The latter result extends Theorem 5.2(3) of Tang & Tsitsiashvili [3] to the case of ultimate ruin. □

REMARK 2.3. Since the asymptotic relations given by Theorems 2.1 and 2.2 are completely explicit, the evaluation of some actuarial quantities becomes quite easy. As an example, we consider the evaluation of stop-loss premiums of the randomly weighted sums (1.1). Under the conditions of Theorem 2.1 with the additional restriction that $\alpha > 1$, we have for each $n = 1, 2, \ldots$ that, as $d \to \infty$,

$$\begin{aligned} \mathrm{E}\left[\sum_{k=1}^{n} \theta_k X_k - d\right]_+ &= \int_d^{\infty} \Pr\left(\sum_{k=1}^{n} \theta_k X_k > x\right) \mathrm{d}x \sim \sum_{k=1}^{n} \mathrm{E}\theta_k^{\alpha} \int_d^{\infty} \overline{F}(x)\mathrm{d}x \\ &= \mathrm{E}[X_1 - d]_+ \sum_{k=1}^{n} \mathrm{E}\theta_k^{\alpha}. \end{aligned}$$

In particular, if the random variables θ_k are given by (2.9) with i.i.d. random variables $\{Y_n, n = 1, 2, \ldots\}$, then for each $n = 1, 2, \ldots$, as $d \to \infty$,

$$\mathrm{E}\left[\sum_{k=1}^{n} \theta_k X_k - d\right]_+ \sim \mathrm{E}[X_1 - d]_+ \frac{\mathrm{E}\, Y_1^{\alpha}(1 - (\mathrm{E}\, Y_1^{\alpha})^n)}{1 - \mathrm{E}\, Y_1^{\alpha}}.$$

Furthermore, if assumption (5) holds, then by Theorem 2.2, it also holds that, as $d \to \infty$,

$$\mathrm{E}\left[\sum_{k=1}^{\infty} \theta_k X_k^+ - d\right]_+ \sim \mathrm{E}[X_1 - d]_+ \frac{\mathrm{E}\, Y_1^{\alpha}}{1 - \mathrm{E}\, Y_1^{\alpha}}. \quad \square$$

3. Some specific cases

In order to apply Theorems 2.1 and 2.2, we need to calculate the expectations $\mathrm{E}\theta_k^{\alpha}$ for $k = 1, 2, \ldots$. In this section, we give some concrete examples in which such calculation can be performed.

3.1. Logelliptically discounted process

It has been pointed out in a large number of papers that the normality assumption regarding log returns of a risky investment is often not realistic. This rejection of normality has led researchers to investigate alternative models for the investment returns, including the family of elliptical distributions. In this direction, we refer the reader to Owen & Rabinovitch [8] and Vorkink [9], among others.

In multivariate statistical analysis, elliptical distributions have provided an alternative to the normal model. Being an extension of the multivariate normal distribution, the class of elliptical distributions shares many of its nice statistical properties, though it contains many other non-normal multivariate distributions such as the multivariate Student's t, Cauchy, logistic, and so on. For details of the class of elliptical distributions, we refer the reader to Fang *et al.* [10] and Gupta *et al.* [11].

There are several equivalent ways to define elliptical distributions. We shall use the definition based on the characteristic function.

DEFINITION 3.1. *A random vector* $\mathbf{Z} = (Z_1, \ldots, Z_n)$ *is said to have an elliptical distribution with parameter vector* μ_n *and parameter matrix*

$$\Sigma_n = \mathbf{B}\mathbf{B}^T$$

for some $n \times m$*-matrix* $\mathbf{B}$*, if its characteristic function is of the form*

$$\mathrm{E}[\exp(i\mathbf{t}'\mathbf{Z})] = \exp(i\mathbf{t}'\mu_n)\phi(\mathbf{t}'\Sigma_n\mathbf{t}),$$

for some function $\phi(\cdot): \mathbb{R} \to \mathbb{R}$*. We write* $\mathbf{Z} =_d E_n(\mu_n, \Sigma_n, \phi)$*. The function* $\phi(\cdot)$ *is called the characteristic generator. The matrix* Σ_n *is symmetric, positive definite and has positive elements on its diagonal.*

The characteristic generator may explicitly depend on n, the dimension of $\mathbf{Z}$. Hence, we denote by Φ_n the family of all possible characteristic generators for a given $n = 1,2,\ldots,$ that is,

$$\Phi_n = \{\phi(\cdot): \phi(t_1^2 + \ldots + t_n^2) \text{ is an } n-\text{dimensional characteristic function}\}.$$

Clearly,

$$\Phi_1 \supset \Phi_2 \supset \Phi_3 \supset \cdots.$$

Let

$$\Phi_\infty = \bigcap_{n=1}^{\infty} \Phi_n.$$

From Theorem 2.21 of Fang *et al.* [10], we know that a function ϕ belongs to the class Φ_∞ if and only if

$$\phi(x) = \int_0^\infty \mathrm{e}^{-xr^2} F_\infty(\mathrm{d}r) \tag{3.1}$$

with F_∞ a distribution function over $(0,\infty)$.

DEFINITION 3.2. *Let* $\mathbf{Y}$ *be a random vector with positive components. We say that* $\mathbf{Y}$ *has a logelliptical distribution with parameters* μ_n*,* Σ_n *and* ϕ*, written as* $\mathbf{Y} =_d LE_n(\mu_n, \Sigma_n, \phi)$*, if*

$$\log \mathbf{Y} = (\log Y_1, \ldots, \log Y_n) =_d E_n(\mu_n, \Sigma_n, \phi).$$

Let us go back to the examples given in Section 1. Recall relation (1.6). We assume that for each $n = 1,2,\ldots,$

$$\mathbf{Z} = (Z_1, \ldots, Z_n) =_d E_n(\mu_n, \Sigma_n, \phi),$$

hence that $\mathbf{Y} = (Y_1, \ldots, Y_n) =_d LE_n(-\mu_n, \Sigma_n, \phi)$. Let σ_{ij}, $i,j = 1, \ldots, n$, denote the entry in row i and column j of the matrix Σ_n. From Theorem 2.16 of Fang *et al.* [10], we know that the marginal distributions and any linear combination of an elliptically distributed random vector are also elliptically distributed with the same generator ϕ. Therefore, with $\mu_{(k)} = \Sigma_{i=1}^{k}\ \mu_i$ and $\sigma_{(k)} = \Sigma_{1 \le i, j \le k}\ \sigma_{ij}$, we have that

$$Z_1 + \cdots + Z_k =_d E_1(\mu_{(k)}, \sigma_{(k)}, \phi),$$

hence that

$$\mathrm{e}^{Z_1 + \cdots + Z_k} =_d LE_1(\mu_{(k)}, \sigma_{(k)}, \phi).$$

Theorem 2.26 of Fang *et al.* [10] gives an explicit expression for the moments $\mathrm{E}\theta_k^{\alpha}$ for $\alpha >$ 0 and $k = 1,2,\ldots$, namely

$$\mathrm{E}\theta_k^{\alpha} = \mathrm{E}\mathrm{e}^{-\alpha(Z_1 + \cdots + Z_k)} = \exp(-\alpha\mu_{(k)})\phi(-\alpha^2\sigma_{(k)}).$$

To consider the infinite dimensional case, we must assume that ϕ is of the form (3.1). In that case, we have that

$$\mathrm{E}\theta_k^{\alpha} = \exp(-\alpha\mu_{(k)}) \int_0^{\infty} \mathrm{e}^{\alpha^2\sigma_{(k)} r^2} F_{\infty}(\mathrm{d}r). \tag{3.2}$$

The lognormally discounted process results when $\phi(x) = \mathrm{e}^{-x/2}$, that is, when the distribution function F_{∞} in (3.1) is degenerated at $1/\sqrt{2}$. Because the lognormal case possesses many attractive properties and is easy to calculate, we restrict ourselves to this case in the remainder of this section. For this case, from (3.2), we have that

$$\mathrm{E}\theta_k^{\alpha} = \exp\left(-\alpha\mu_{(k)} + \frac{1}{2}\alpha^2\sigma_{(k)}\right). \tag{3.3}$$

Therefore, under the conditions of Theorems 2.1 and 2.2, we have that for $n = 1,2,\ldots$

$$\Pr\left(\sum_{k=1}^{n}\ \theta_k X_k > x\right) \sim \overline{F}(x) \sum_{k=1}^{n}\ \exp\left(-\alpha\mu_{(k)} + \frac{1}{2}\alpha^2\sigma_{(k)}\right). \tag{3.4}$$

Now we give some numerical results for relation (3.4). We assume that the random variables $\{X_n, n = 1, 2, \ldots\}$ are i.i.d. with common Pareto(α,β) distribution for some $\alpha > 0$ and $\beta > 0$, with density function

$$f_X(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \quad x > \beta,$$

and that for each $n = 1, 2, \ldots$, the vector $(Y_1, \ldots Y_n)$ follows an n-dimensional lognormal distribution with parameters $-\mu_n, \Sigma_n$. We take the dimension $n = 10$, the mean vector $\mu_{10} = (0.1, 0.1, \ldots, 0.1)$ and the covariance matrix

$$\Sigma_{10} = \begin{pmatrix} 0.05 & 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.1 & 0.01 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0.1 & 0.01 & 0.02 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0.01 & 0.05 & 0.05 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & 0.05 & 0.1 & 0.01 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0.01 & 0.1 & 0.02 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0.02 & 0.05 & 0.01 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 & 0.01 & 0.02 & 0.01 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0.01 & 0.1 & 0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0.05 & 0.05 \end{pmatrix}.$$

Some numerical results are given in Table 1. The number of simulations is 5,000,000. The considered values of α are 1.2 and 1.5 as is reasonable, for example, in fire insurance; see Beirlant *et al.* [12]. Apart from the values of x and the simulated and asymptotic tail probabilities in (3.4), we also display the values of $1 - \frac{\text{asymptotic}}{\text{simulated}}$. Theoretically, these values must tend to 0 when $x \to \infty$. This seems to be the case from the table.

Table 2 presents some numerical results for the case of $n = 50$, i.e., when more time periods are considered. The values of the elements of μ_{50} and Σ_{50} are similar to the ones for $n = 10$. The number of simulations is again 5,000,000.

In the notes of the tables, we display the simulated values of several quantiles of the discounted sums under consideration.

3.2. Lognormal variance-mean mixed discounted process

As announced in Example 1.3, we now concentrate on the situation when the vector (1.6) is a normal variance-mean mixture with some mixing law. Some distributions from this

Table 1. Simulated versus asymptotic values of the tail probability for Pareto claims with lognormal discount factors ($n = 10$)

x	$\alpha = 1.2$, Simulated	$\beta = 2$ Asymptotic	$1 - \frac{\text{asymptotic}}{\text{simulated}}$	x	$\alpha = 1.5$, Simulated	$\beta = 2$ Asymptotic	$1 - \frac{\text{asymptotic}}{\text{simulated}}$
300	0.03091	0.02051	0.337	100	0.08002	0.02631	0.671
400	0.02010	0.01452	0.278	200	0.01976	0.00930	0.529
500	0.01451	0.01111	0.234	300	0.00869	0.00506	0.417
600	0.01117	0.00893	0.201	400	0.00500	0.00329	0.342
700	0.00901	0.00742	0.177	500	0.00326	0.00235	0.278
800	0.00747	0.00632	0.154	600	0.00237	0.00179	0.244
900	0.00638	0.00549	0.140	700	0.00179	0.00142	0.207
1000	0.00551	0.00484	0.122	800	0.00141	0.00116	0.178
1500	0.00326	0.00297	0.088	900	0.00116	0.00097	0.159
2000	0.00222	0.00210	0.052	1000	0.00096	0.00083	0.136
2500	0.00164	0.00161	0.018	1500	0.00050	0.00045	0.093
3000	0.00135	0.00129	0.040	2000	0.00032	0.00029	0.079
3500	0.00111	0.00108	0.034	2500	0.00023	0.00021	0.078
4000	0.00094	0.00092	0.028	3000	0.00017	0.00016	0.058
4500	0.00082	0.00080	0.026	3500	0.00013	0.00013	0.023
5000	0.00072	0.00070	0.021	4000	0.00011	0.00010	0.010

Notes: Simulated quantiles at level p for $\alpha = 1.2$ are as follows: 219.65 ($p = 0.950$), 345.18 ($p = 0.975$), 649.21 ($p = 0.990$), 1083.0 ($p = 0.995$), 3818.6 ($p = 0.999$). Simulated quantiles at level p for $\alpha = 1.5$ are as follows: 126.90 ($p = 0.950$), 178.28 ($p = 0.975$), 279.87 ($p = 0.990$), 400.05 ($p = 0.995$), 981.18 ($p = 0.999$).

Table 2. Simulated versus asymptotic values of the tail probability for Pareto claims with lognormal discount factors ($n = 50$)

x	$\alpha = 1.2$, Simulated	$\beta = 2$ Asymptotic	$1 - \frac{\text{asymptotic}}{\text{simulated}}$	x	$\alpha = 1.5$, Simulated	$\beta = 2$ Asymptotic	$1 - \frac{\text{asymptotic}}{\text{simulated}}$
500	0.0791840	0.0343533	0.566	500	0.0377720	0.0104716	0.723
1000	0.0316480	0.0149532	0.528	1000	0.0127510	0.0037023	0.710
2500	0.0088520	0.0049797	0.437	2500	0.0027794	0.0009366	0.663
5000	0.0033138	0.0021675	0.346	5000	0.0008460	0.0003311	0.609
7500	0.0018802	0.0013325	0.291	7500	0.0004206	0.0001803	0.571
10000	0.0012676	0.0009435	0.256	10000	0.0002510	0.0001171	0.534
15000	0.0007298	0.0005800	0.205	15000	0.0001228	0.0000637	0.481
20000	0.0005034	0.0004107	0.184	20000	0.0000692	0.0000414	0.402
25000	0.0003746	0.0003142	0.161	25000	0.0000466	0.0000296	0.364
30000	0.0002964	0.0002525	0.148	30000	0.0000348	0.0000225	0.353
40000	0.0002032	0.0001788	0.120	35000	0.0000294	0.0000179	0.392
50000	0.0001528	0.0001368	0.105	40000	0.0000218	0.0000146	0.329
60000	0.0001222	0.0001099	0.101	45000	0.0000184	0.0000123	0.333
70000	0.0000980	0.0000913	0.068	50000	0.0000148	0.0000105	0.292
80000	0.0000818	0.0000778	0.049	60000	0.0000094	0.0000080	0.153
90000	0.0000702	0.0000676	0.038	70000	0.0000074	0.0000063	0.146
100000	0.0000616	0.0000595	0.034	80000	0.0000056	0.0000052	0.076

Notes: Simulated quantiles at level p for $\alpha = 1.2$ are as follows: 711.18 ($p = 0.950$), 1187.3 ($p = 0.975$), 2292.9 ($p = 0.990$), 3733.6 ($p = 0.995$), 11931 ($p = 0.999$). Simulated quantiles at level p for $\alpha = 1.5$ are as follows: 414.11 ($p = 0.950$), 654.68 ($p = 0.975$), 1164.5 ($p = 0.990$), 1773.6 ($p = 0.995$), 4546.1 ($p = 0.999$).

class have already been studied in the financial literature; see Eberlein & Keller [13], Barndorff-Nielsen [14] and Bingham *et al.* [6].

Definition 3.3. *A random vector* $\mathbf{Z} = (Z_1, \ldots, Z_n)$ *is said to be a normal variance-mean mixture with position* μ_n, *drift* β_n, *structure matrix* $\mathbf{\Sigma}_n$ *and mixing distribution* G *on* $[0, \infty)$ *if for some random variable* U *sampled from* G, *the conditional distribution of* $\mathbf{Z}$ *given* $(U = u)$ *is* $N_n(\mu_n + u\beta_n, u\mathbf{\Sigma}_n)$. *Here the structure matrix* $\mathbf{\Sigma}_n$ *is symmetric and positive definite with* $|\mathbf{\Sigma}_n| = 1$. *We write* $\mathbf{Z} =_d NVMM_n(\mu_n, \beta_n, \mathbf{\Sigma}_n, G)$.

The characteristic function of $\mathbf{Z}$ is then given by

$$\varphi_{\mathbf{Z}}(\mathbf{t}) = \exp(i\mathbf{t}'\mu_n)\Phi\left(\frac{1}{2}\mathbf{t}'\Sigma_n\mathbf{t} - i\mathbf{t}'\beta_n\right),$$

where Φ is the Laplace-Stieltjes transform of G, that is,

$$\Phi(s) = \int_0^\infty \mathrm{e}^{-sr} G(\mathrm{d}r), \quad s > 0.$$

Definition 3.4. *If* $\beta_n = 0$ *in the above definition, then we obtain the class of normal variance mixtures, denoted* $NVM\ (\mu_n, \mathbf{\Sigma}_n, G)$.

In fact, in this case we have $\mathbf{Z} =_d E_n(\mu_n, \mathbf{\Sigma}_n, \phi)$ with $\phi(s) = \Phi(s/2)$, so that $NVM_n \subset E_n$; see Bingham *et al.* [6].

Definition 3.5. *Let* $\mathbf{Y}$ *be a random vector with positive components. We say that* $\mathbf{Y}$ *has a lognormal variance-mean mixed distribution with parameters* μ_n, β_n, Σ_n *and* G, *denoted* $LNVMM_n$, *if*

$$\log\mathbf{Y} = (\log Y_1, \ldots, \log Y_n) =_d NVMM_n(\mu_n, \beta_n, \Sigma_n, G).$$

Let us go back again to the examples given in Section 1. We assume that $\mathbf{Z} = (Z_1, ..., Z_n) =_d NVMM_n(\mu_n, \beta_n, \mathbf{\Sigma}_n, G)$. From Definition 3.3, for $k = 1, 2, \ldots$ and $u > 0$, we have

$$(Z_1 + ... + Z_k)|(U = u) =_d N_1\left(\sum_{i=1}^{k} (\mu_i + u\beta_i), u\sigma_{(k)}\right),$$

where $\sigma_{(k)} = \Sigma_{1 \le i,j \le k}\ \sigma_{ij}$ as before. It follows that for $\alpha > 0$,

$$\begin{aligned} \mathrm{E}\theta_k^{\alpha} &= \mathrm{E}[\mathrm{E}(\mathrm{e}^{-\alpha(Z_1+\ldots+Z_k)}|\,U)] \\ &= \mathrm{E}\left[\exp\left(-\alpha\sum_{i=1}^{k} (\mu_i + \beta_i U) + \frac{\alpha^2\sigma_{(k)}}{2}U\right)\right] \\ &= \exp\left(-\alpha\sum_{i=1}^{k} \mu_i\right)\Phi\left(\alpha\sum_{i=1}^{k} \beta_i - \frac{\alpha^2\sigma_{(k)}}{2}\right). \end{aligned}$$

This gives an explicit expression for the moments $\mathrm{E}\theta_k^{\alpha}$ for $\alpha > 0$ and $k = 1, 2, \ldots$. Therefore, under the conditions of Theorems 2.1 and 2.2, we have in this case that for $n = 1, 2, \ldots, \infty$,

$$\Pr\left(\sum_{k=1}^{n} \theta_k X_k > x\right) \sim \overline{F}(x)\sum_{k=1}^{n} \exp\left(-\alpha\sum_{i=1}^{k} \mu_i\right)\Phi\left(\alpha\sum_{i=1}^{k} \beta_i - \frac{\alpha^2\sigma_{(k)}}{2}\right). \tag{3.5}$$

In particular, when $\beta_n = 0$ and $\Phi(s) = \mathrm{e}^{-s}$, we are again in the lognormal setting and relation (3.5) coincides with relation (3.4).

In the following, we consider the particular case when $\mathbf{Y}$ is a lognormal variance-mean mixture with the inverse Gaussian distribution as the mixing distribution. This mixing distribution was also considered e.g., by Barndorff-Nielsen [14] and Bingham *et al.* [6]. The inverse Gaussian density is

$$g(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda}{2v^2 x}(x - v)^2\right), \quad x > 0,$$

with $\lambda, v > 0$, and its Laplace-Stieltjes transform is

$$\Phi(s) = \exp\left(\frac{\lambda}{v}\left(1 - \sqrt{1 + \frac{2sv^2}{\lambda}}\right)\right), \quad s > 0.$$

Hence, from (3.5), in this case

$$\Pr\left(\sum_{k=1}^{n} \theta_k X_k > x\right) \sim \overline{F}(x)\sum_{k=1}^{n} \exp\left(-\alpha\mu_{(k)} + \frac{\lambda}{v}\left(1 - \sqrt{1 + \frac{2\alpha v^2}{\lambda}\left(\sum_{i=1}^{k} \beta_i - \frac{\alpha\sigma_{(k)}}{2}\right)}\right)\right).$$

To assess the quality of this asymptotic relation by simulation, we assume, as in the previous section, that the random variables $\{X_n, n = 1, 2, \ldots\}$ are i.i.d. with common

Table 3. Simulated versus asymptotic values of the tail probability for Pareto claims with lognormal variance-mean inverse Gaussian mixed discount factors

x	$\alpha=1.2$, Simulated	$\beta=2$ Asymptotic	$1-\frac{\text{asymptotic}}{\text{simulated}}$	x	$\alpha=1.5$, Simulated	$\beta=2$ Asymptotic	$1-\frac{\text{asymptotic}}{\text{simulated}}$
100	0.0097178	0.0082910	0.146	100	0.0022460	0.0019209	0.144
200	0.0039028	0.0036089	0.075	200	0.0007368	0.0006791	0.078
300	0.0023458	0.0022185	0.054	300	0.0003892	0.0003696	0.050
400	0.0016464	0.0015708	0.045	400	0.0002484	0.0002401	0.033
500	0.0012508	0.0012018	0.039	500	0.0001772	0.0001718	0.030
600	0.0010012	0.0009656	0.035	600	0.0001358	0.0001307	0.037
700	0.0008350	0.0008025	0.038	700	0.0001070	0.0001037	0.030
800	0.0007038	0.0006837	0.028	800	0.0000848	0.0000848	−0.001
900	0.0006108	0.0005936	0.028	900	0.0000718	0.0000711	0.009
1000	0.0005410	0.0005231	0.033	1000	0.0000616	0.0000607	0.013
2000	0.0002338	0.0002277	0.026	1200	0.0000450	0.0000462	−0.026
3000	0.0001428	0.0001399	0.019	1400	0.0000356	0.0000366	−0.030
4000	0.0000992	0.0000991	0.001	1800	0.0000244	0.0000251	−0.030

Pareto (α,β) distribution, for some $\alpha>0$ and $\beta>0$. We take again $n=10$ and consider the parameters $-\mu_{10}, \Sigma_{10}$ of the lognormal variance-mean inverse Gaussian mixture to be as given before, while

$$\beta_1=\cdots=\beta_{10}=1, \quad \lambda=1 \text{ and } v=1.$$

The number of simulations is 5,000,000. Some numerical results are given in Table 3.

4. Proof of the theorems

4.1. Some lemmas

The following lemma is from Breiman [15]; see also Cline & Samorodnitsky [16] for more general discussions.

LEMMA 4.1. *Let X and Y be two independent random variables with cdf's F and G, where Y is nonnegative. If* $F\in\mathcal{R}_{-\alpha}$ *for some* $\alpha>0$ *and* $\mathrm{E}\,Y^{\alpha+\delta}<\infty$ *for some* $\delta>0$, *then*

$$\lim_{x\to\infty}\frac{\Pr(XY>x)}{\Pr(X>x)}=\mathrm{E}\,Y^{\alpha}.$$

The following is a restatement of Lemma 2.1 of Davis & Resnick [17].

LEMMA 4.2. *For a sequence of nonnegative random variables* $\{X_1,\ldots,X_n\}$ *and a distribution function* $F\in\mathcal{R}_{-\alpha}$ *for some* $\alpha>0$, *if*

$$\lim_{x\to\infty}\frac{\Pr(X_i>x)}{\overline{F}(x)}=c_i \quad \text{for } i=1,\ldots,n$$

and

$$\lim_{x\to\infty}\frac{\Pr(X_i>x,X_j>x)}{\overline{F}(x)}=0 \quad \text{for } 1\le i\ne j\le n,$$

then

$$\lim_{x\to\infty}\frac{\Pr\left(\sum_{i=1}^{n} X_i > x\right)}{\overline{F}(x)}=\sum_{i=1}^{n} c_i.$$

The following result is the one-dimensional version of Theorem 2.1 of Resnick & Willekens [1].

LEMMA 4.3. *Consider the randomly weighted series* $\Sigma_{n=1}^{\infty}\theta_n X_n^+$ *with* $\{X_n,\ n=1,2,\ldots\}$ *and* $\{\theta_n, n=1,2,\ldots\}$ *the same as in (1.1). Then under the conditions of Theorem 2.2, we have*

$$\Pr\left(\sum_{n=1}^{\infty}\theta_n X_n^+ > x\right)\sim\overline{F}(x)\sum_{n=1}^{\infty}\mathrm{E}\theta_n^{\alpha}.$$

4.2. Proof of Theorem 2.1

By Lemma 4.1, we have

$$\Pr(\theta_k X_k^+ > x)\sim\overline{F}(x)\mathrm{E}\theta_k^{\alpha},\quad 1\le k\le n.$$

Now choose some $0<\varepsilon<1$ such that $(1-\varepsilon)(\alpha+\delta)>\alpha$. Then for all $1\le k\ne l\le n$,

$$\begin{aligned}\Pr(\theta_k X_k^+ > x,\theta_l X_l^+ > x)&\le\Pr(\theta_k > x^{1-\varepsilon})+\Pr(\theta_k X_k^+ > x,\theta_l X_l^+ > x,\theta_k\le x^{1-\varepsilon})\\&\le x^{-(1-\varepsilon)(\alpha+\delta)}\mathrm{E}\theta_k^{\alpha+\delta}+\Pr(X_k^+ > x^{\varepsilon})\Pr(\theta_l X_l^+ > x)\\&=o(\overline{F}(x)),\end{aligned}$$

where we have used the Markov inequality and the property in (2.2). Thus, applying Lemma 4.2 we obtain that

$$\Pr\left(\sum_{k=1}^{n}\theta_k X_k^+ > x\right)\sim\overline{F}(x)\sum_{k=1}^{n}\mathrm{E}\theta_k^{\alpha}.\tag{4.1}$$

Since

$$\sum_{k=1}^{n}\theta_k X_k\le\max_{1\le m\le n}\sum_{k=1}^{m}\theta_k X_k\le\sum_{k=1}^{n}\theta_k X_k^+,$$

it suffices to prove the relation

$$\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x\right)\gtrsim\overline{F}(x)\sum_{k=1}^{n}\mathrm{E}\theta_k^{\alpha}.\tag{4.2}$$

For an arbitrary set $\mathcal{I}\subset\{1,2,\cdots,n\}$, we denote by $\|\mathcal{I}\|$ the cardinal number of the set $\mathcal{I}$ and introduce two events

$$\Omega_1(\mathcal{I})=(X_k>0\ \text{ for all }\ k\in\mathcal{I}),\quad\Omega_2(\mathcal{I})=(X_k\le 0\ \text{ for all }\ k\notin\mathcal{I}).$$

Clearly,

$$\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x\right) = \sum_{\mathcal{I}:\mathcal{I}\subset\{1,2,\cdots,n\}\&\mathcal{I}\neq\phi} \Pr\left(\sum_{k=1}^{n}\theta_k X_k > x, \Omega_1(\mathcal{I})\cap\Omega_2(\mathcal{I})\right). \tag{4.3}$$

For any large $L > 0$ and $M > 0$, we further write

$$\Omega_3(\mathcal{I};L) = (-L < X_k \le 0 \;\text{ for }\; k\notin\mathcal{I}), \quad \Omega_4(\mathcal{I};M) = (\theta_k \le M \;\text{ for }\; k\notin\mathcal{I}).$$

Then, the probability on the right-hand side of (4.3) is not smaller than

$$\begin{aligned}
&\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x, \Omega_1(\mathcal{I})\cap\Omega_2(\mathcal{I})\cap\Omega_3(\mathcal{I};L)\cap\Omega_4(\mathcal{I};M)\right)\\
&\ge \Pr\left(\sum_{k\in I}\theta_k X_k > x + (n-\|\mathcal{I}\|)LM, \Omega_1(\mathcal{I})\cap\Omega_3(\mathcal{I};L)\cap\Omega_4(\mathcal{I};M)\right)\\
&= \Pr\left(\sum_{k\in\mathcal{I}}\theta_k X_k > x + (n-\|\mathcal{I}\|)LM \,\middle|\, \Omega_1(\mathcal{I})\cap\Omega_3(\mathcal{I};L)\cap\Omega_4(\mathcal{I};M)\right)\\
&\quad\times\Pr(\Omega_1(\mathcal{I}))\Pr(\Omega_3(\mathcal{I};L))\Pr(\Omega_4(\mathcal{I};M)).
\end{aligned} \tag{4.4}$$

Similarly to (4.1), applying Lemma 4.2, the conditional probability on the right-hand side of (4.4) is asymptotically equal to

$$\sum_{k\in\mathcal{I}} \mathrm{E}(\theta_k^{\alpha}|\Omega_4(\mathcal{I};M))\Pr(X_k > x + (n-\|\mathcal{I}\|)LM | X_k > 0).$$

Since $F\in\mathcal{R}$ implies

$$\Pr(X_k > x + (n-\|\mathcal{I}\|)LM|X_k > 0) \sim \Pr(X_k > x|X_k > 0),$$

we have

$$\begin{aligned}
&\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x, \Omega_1(\mathcal{I})\cap\Omega_2(\mathcal{I})\cap\Omega_3(\mathcal{I};L)\cap\Omega_4(\mathcal{I};M)\right)\\
&\gtrsim \frac{\overline{F}(x)}{\overline{F}(0)}\sum_{k\in\mathcal{I}} \mathrm{E}(\theta_k^{\alpha}|\Omega_4(\mathcal{I};M))\Pr(\Omega_1(\mathcal{I}))\Pr(\Omega_3(\mathcal{I};L))\Pr(\Omega_4(\mathcal{I};M))\\
&= \frac{\overline{F}(x)}{\overline{F}(0)}\sum_{k\in\mathcal{I}} \mathrm{E}\theta_k^{\alpha}1_{\Omega_4(\mathcal{I};M)}\Pr(\Omega_1(\mathcal{I}))\Pr(\Omega_3(\mathcal{I};L)).
\end{aligned}$$

Substituting this into (4.3) yields that

$$\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x\right) \gtrsim \frac{\overline{F}(x)}{\overline{F}(0)} \sum_{\mathcal{I}:\mathcal{I}\subset\{1,2,\cdots,n\}\&\mathcal{I}\neq\phi}\;\sum_{k\in\mathcal{I}} \mathrm{E}\theta_k^{\alpha}1_{\Omega_4(\mathcal{I};M)}\Pr(\Omega_1(\mathcal{I}))\Pr(\Omega_3(\mathcal{I};L)).$$

Since L and M can be arbitrarily large, this proves that

$$\Pr\left(\sum_{k=1}^{n}\theta_k X_k > x\right) \gtrsim \frac{\overline{F}(x)}{\overline{F}(0)} \sum_{\mathcal{I}:\mathcal{I}\subset\{1,2,\cdots,n\}\&\mathcal{I}\neq\phi}\;\sum_{k\in\mathcal{I}} \mathrm{E}\theta_k^{\alpha}(\overline{F}(0))^{\|\mathcal{I}\|}(F(0))^{n-\|\mathcal{I}\|}.$$

By interchanging the order of the two sums, we calculate the right-hand side of the above as

$$\frac{\overline{F}(x)}{\overline{F}(0)}\sum_{k=1}^{n}\ \mathrm{E}\theta_k^{\alpha}\sum_{\mathcal{I}:\mathcal{I}\subset\{1,2,\cdots,n\}\&k\in\mathcal{I}}(\overline{F}(0))^{\|\mathcal{I}\|}(F(0))^{n-\|\mathcal{I}\|}=\frac{\overline{F}(x)}{\overline{F}(0)}\sum_{k=1}^{n}\ \mathrm{E}\theta_k^{\alpha}(\overline{F}(0)+F(0))^{n-1}\overline{F}(0).$$

This proves the announced result (4.2).

4.3. Proof of Theorem 2.2

It is trivial that

$$\Pr\left(\max_{1\le n<\infty}\sum_{k=1}^{n}\ \theta_k X_k>x\right)\le\Pr\left(\sum_{k=1}^{\infty}\ \theta_k X_k^+>x\right).$$

In view of this, relation (2.5), and the convergence of the series $\Sigma_{k=1}^{\infty}\ \mathrm{E}\theta_k^{\alpha}$, it suffices to prove that

$$\Pr\left(\sum_{k=1}^{\infty}\ \theta_k X_k^+>x\right)\sim\overline{F}(x)\sum_{k=1}^{\infty}\ \mathrm{E}\theta_k^{\alpha}. \tag{4.5}$$

However, relation (4.5) is given by Lemma 4.3.

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