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Worst VaR scenarios: A remark

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ABSTRACT

Theorem 15 of Embrechts et al. [Embrechts, Paul, Höing, Andrea, Puccetti, Giovanni, 2005. Worst VaR scenarios. Insurance: Math. Econom. 37, 115–134] proves that comonotonicity gives rise to the *on-average-most-adverse* Value-at-Risk scenario for a function of dependent risks, when the marginal distributions are known but the dependence structure between the risks is unknown. This note extends this result to the case where, rather than no information, partial information is available on the dependence structure between the risks. A result of Kaas et al. [Kaas, Rob, Dhaene, Jan, Goovaerts, Marc J., 2000. Upper and lower bounds for sums of random variables. Insurance: Math. Econom. 23, 151–168] is also generalized for this purpose.

1. Introduction

Finding best-possible upper bounds on the Value-at-Risk (VaR) for a function of dependent risks when the marginal distributions of the risks are known but the dependence structure between the risks is not or only partially known has been an important research topic for many years. Contributions include – without being exhaustive – Makarov (1981), Rüschendorf (1982), Frank et al. (1987), Denuit et al. (1999, 2005), Embrechts and Puccetti (2006) and Kaas et al. (under review).

These works have revealed that in general the *comonotonic* dependence structure does not lead to the *worst VaR scenario*. At first sight, this may seem surprising: comonotonicity, under which every risk is a non-decreasing function of a common risk factor, is generally perceived to be the strongest dependence notion. It is the worst possible dependence scenario in the sense of *stop-loss* and *supermodular* ordering.

Nevertheless, as becomes apparent, for example, from Theorem 6 of Embrechts et al. (2005), there exists in general for any probability level a copula that yields a VaR that is larger than the VaR under comonotonicity. The reader is referred to Dhaene et al. (2002a,b) for an extensive study of the concept of comonotonicity and its applications in insurance and finance.

Though comonotonicity may not lead to the worst VaR scenario *for a given probability level*, Theorem 15 of Embrechts et al. (2005) proves that the comonotonic dependence structure does give rise to the VaR scenario that is most adverse *on average* for a function of dependent risks; a precise statement of this result is deferred until Section 2.

The aim of this note is to extend this result to the case where, instead of no information, partial information is available on the dependence structure between the risks. In particular, I assume that there exists a common risk factor with a given distribution function conditionally upon which the marginal distribution functions of the risks are available. I then prove that the *improved comonotonic* (or *conditionally comonotonic*) dependence structure as introduced in Kaas et al. (2000) – see also Dhaene et al. (2002a) – arises as the on-average-most-adverse VaR scenario. It supports

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the use of comonotonic and conditionally comonotonic scenarios *also* in VaR-based risk management.[1]

The outline of this note is as follows: in Section 2, I introduce some preliminaries on worst VaR scenarios. In Section 3, I recall the improved comonotonic dependence structure and derive some new results for it. Section 4 contains the main results and Section 5 concludes.

2. Worst VaR scenarios: Preliminaries

I fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a random vector $\underline{X} := (X_1, \ldots, X_n)$ defined on it. For a given measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$, I consider the problem of finding the best-possible lower bound on the probability

$$\mathbb{P}[\psi(\underline{X}) < s], \quad s \in \mathbb{R}, \tag{1}$$

when the marginal distribution functions (d.f.'s) $F_{X_1}, \ldots, F_{X_n}$ of $\underline{X}$ are known but its dependence structure is unknown.

Equivalently, given the marginal d.f.'s of $\underline{X}$, one can consider the problem of finding the best-possible upper bound on

$$\mathrm{VaR}_\alpha[\psi(\underline{X})], \quad \alpha \in (0, 1). \tag{2}$$

Here, as usual, the VaR at probability level α is defined by $\mathrm{VaR}_\alpha[X] := F_X^{-1}(\alpha)$ with

$$F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} | F_X(x) \geq \alpha\}.$$

Notice that $\mathrm{VaR}_\alpha[X]$ is a left-continuous function of α.

I denote by $C : [0, 1]^n \to [0, 1]$ an n-copula and by $\mathcal{C}^n$ the family of all n-copulas. Furthermore, I denote by $M(u_1, \ldots, u_n) := \min\{u_1, \ldots, u_n\}$, the Fréchet upper bound. For any positive integer n, $M \in \mathcal{C}^n$. The reader is referred to Nelsen (1999) for further details on copulas. In the following, given the marginals $F_1, \ldots, F_n$, let $\underline{X}^C$ denote the random vector induced by the n-copula C.

For $n = 2$, the copula that gives rise to the best-possible lower bound on the probability in (1) is known; see, for example, Theorem 3.1 of Embrechts and Puccetti (2006). It is known that in general this copula depends on the value of s and hence, in VaR terms, on the probability level α. Clearly, this is highly inconvenient from a practical point of view.

Noting this, Embrechts et al. (2005) introduced, for a given copula C, the loss function

$$e_{C,\psi}(s) := \mathbb{P}[\psi(\underline{X}^C) < s] - \inf_{\tilde{C} \in \mathcal{C}^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}, \tag{3}$$

and considered the following optimization problem:

$$\inf_{C \in \mathcal{C}^n} \left\{ \int_d^{+\infty} e_{C,\psi}(s) \mathrm{d}s \right\}. \tag{4}$$

Now, I restate Theorem 15 of Embrechts et al. (2005); for a definition of supermodularity, I refer the reader to Denuit et al. (2005, p. 179).

Lemma 2.1. *For every real number d and every non-decreasing supermodular function ψ satisfying $\mathbb{E}[\psi(\underline{X}^M)] < +\infty$, M is a minimizer of (4).*

Remark 2.1. Note that (4) can be solved for any $n \geq 2$ even though when $n > 2$ the copula that gives rise to the best-possible lower bound on the second probability on the right-hand side of (3) is (as yet) unknown.

[1] In contrast to the pervasive VaR, all *concave distortion risk measures* (which include, most noticeably, the Tail-Value-at-Risk, and are sometimes referred to as *spectral risk measures*) assume their worst-case scenario when the risks are comonotonic; see Kaas et al. (under review) and Section 3. Therefore, comonotonic scenarios already play a clear-cut role in risk management based on concave distortion risk measures.

3. Additional information on the dependence structure

In the remainder of this paper I assume that there exists a random variable (r.v.) Λ with a given d.f. such that, conditionally upon $\Lambda = \lambda$, the conditional marginal d.f.'s $F_{X_1|\Lambda=\lambda}, \ldots, F_{X_n|\Lambda=\lambda}$ of $\underline{X}$ are available (objectively known), for all $\lambda \in \mathrm{supp}\Lambda$.

Consider the following simple but already practically relevant example:

Example 3.1. Let $S := X_1 + \cdots + X_n$ represent the stochastically discounted sum of running year losses of n portfolios that an insurer holds. Let us assume that every X_i can be decomposed into a (usually unhedgeable) "insurance risk" Y_i and a common (usually hedgeable) "financial risk" Λ, such that for all $i = 1, \ldots, n$, $(X_i|\Lambda = \lambda) \overset{\text{a.s.}}{=} \lambda Y_i$. Suppose that Λ with given d.f. F_Λ is independent of $\underline{Y}$ with given marginal d.f.'s $F_{Y_1}, \ldots, F_{Y_n}$, and that the dependence structure within $\underline{Y}$ is unknown. The aim is to determine a lower bound on $\mathbb{P}[S < s]$.

The reader easily verifies that Example 3.1 is compatible with the assumptions of the partial information model outlined above: knowledge of the marginals $F_{X_1}, \ldots, F_{X_n}$, of the conditional marginals $F_{X_1|\Lambda=\lambda}, \ldots, F_{X_n|\Lambda=\lambda}$, $\lambda \in \mathrm{supp}\Lambda$, and of the d.f. F_Λ. It is important to note that while the functional dependence linking the portfolio risks to the common risk factor Λ is (almost surely) fully specified in this example, the assumptions of the present set-up are much weaker because they do not require a pre-specified functional dependence relation.

Next, the notions of *stop-loss* and *supermodular* order are introduced. I say that a r.v. X is smaller than a r.v. Y in *stop-loss* order if for any non-decreasing and convex function $f : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)],$$

provided that the expectations exist. Furthermore, I say that a random vector $\underline{X}$ is smaller than a random vector $\underline{Y}$ in *supermodular* order if for any supermodular function $f : \mathbb{R}^n \to \mathbb{R}$

$$\mathbb{E}[f(\underline{X})] \leq \mathbb{E}[f(\underline{Y})],$$

provided that the expectations exist. I shall write $X \leq_{\mathrm{sl}} Y$ and $\underline{X} \leq_{\mathrm{sm}} \underline{Y}$, respectively. As is well-known (see, for example, Proposition 6.3.7 of Denuit et al. (2005)), for all $C \in \mathcal{C}^n$,

$$\underline{X}^C \leq_{\mathrm{sm}} \underline{X}^M. \tag{5}$$

Then, I state the following theorem, which extends Proposition 2 of Kaas et al. (2000) where only the case of $\psi = +$ is considered:

Theorem 3.1. *Let U be a r.v. uniformly distributed on $(0, 1)$ and independent of Λ. Then, for any non-decreasing supermodular function $\psi : \mathbb{R}^n \to \mathbb{R}$,*

$$\begin{aligned} \psi(X_1, \ldots, X_n) &\leq_{\mathrm{sl}} \psi(F_{X_1|\Lambda}^{-1}(U), \ldots, F_{X_n|\Lambda}^{-1}(U)) \\ &\leq_{\mathrm{sl}} \psi(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)). \end{aligned} \tag{6}$$

Proof. It is well-known that for any non-decreasing convex function v, $v \circ \psi$ is supermodular; see, for example, Proposition 3.4.67 of Denuit et al. (2005). Hence, recalling (5),

$$\begin{aligned} &\mathbb{E}[v \circ \psi(X_1, \ldots, X_n)] \\ &= \int_{\mathrm{supp}\Lambda} \mathbb{E}[v \circ \psi(X_1, \ldots, X_n) | \Lambda = \lambda] \mathrm{d}F_\Lambda(\lambda) \\ &\leq \int_{\mathrm{supp}\Lambda} \mathbb{E}[v \circ \psi(F_{X_1|\Lambda=\lambda}^{-1}(U), \ldots, F_{X_n|\Lambda=\lambda}^{-1}(U))] \mathrm{d}F_\Lambda(\lambda) \\ &= \mathbb{E}[v \circ \psi(F_{X_1|\Lambda}^{-1}(U), \ldots, F_{X_n|\Lambda}^{-1}(U))]. \end{aligned}$$

This proves the first inequality in (6). Furthermore, since $(F_{X_1|\Lambda}^{-1}(U), \ldots, F_{X_n|\Lambda}^{-1}(U))$ has marginals $F_{X_1}, \ldots, F_{X_n}$ it also follows from (5) and the supermodularity of $v \circ \psi$ that

$$\mathbb{E}[v \circ \psi(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U))] \leq \mathbb{E}[v \circ \psi(F^{-1}_{X_1}(U), \dots, F^{-1}_{X_n}(U))].$$

This proves the second inequality in (6). □

In the present set-up, the dependence structure within the random vector

$$(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U))$$

is referred to as the *improved comonotonic* or *conditionally comonotonic* dependence structure.

Corollary 3.1. *Recalling that concave distortion risk measures (including the Tail-Value-at-Risk as popular alternative to the pervasive VaR) preserve stop-loss order (see Denuit et al. (2005), Proposition 3.4.7, and Dhaene et al. (2008)) it follows that for any non-decreasing supermodular function ψ and any concave distortion risk measure π,*

$$\pi[\psi(X_1, \dots, X_n)] \leq \pi[\psi(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U))] \leq \pi[\psi(F^{-1}_{X_1}(U), \dots, F^{-1}_{X_n}(U))]. \quad (7)$$

Remark 3.1. Notice that, under the assumptions of the partial information model, for any non-decreasing supermodular function ψ,

$$\psi(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U))$$

is the **best-possible** upper bound on $\psi(\underline{X})$ in stop-loss order sense.

Remark 3.2. Notice that:
if $\underline{X}$ were comonotonic,

$$\underline{X} \stackrel{d}{=} (F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U)) \stackrel{d}{=} (F^{-1}_{X_1}(U), \dots, F^{-1}_{X_n}(U));$$

if $\underline{X}$ and Λ were independent,

$$(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U)) = (F^{-1}_{X_1}(U), \dots, F^{-1}_{X_n}(U)).$$

Example 3.2 (*Continuation of* Example 3.1). Under the assumptions of Example 3.1, Theorem 3.1 implies that

$$S \leq_{sl} \sum_{i=1}^{n} F^{-1}_{X_i|\Lambda}(U) \stackrel{d}{=} F^{-1}_{\Lambda}(V) \sum_{i=1}^{n} F^{-1}_{Y_i}(U) \leq_{sl} \sum_{i=1}^{n} F^{-1}_{X_i}(U),$$

where the r.v.'s U, V are uniformly distributed on (0, 1), mutually independent and independent of Λ.

Example 3.3. Let the r.v.'s X_1, X_2 and Λ be uniformly distributed on (0, 1). Assume that $(X_1|\Lambda = \lambda) \stackrel{\text{a.s.}}{=} \lambda$ and $(X_2|\Lambda = \lambda) \stackrel{\text{a.s.}}{=} 1 - \lambda$. Then

$$(F^{-1}_{X_1|\Lambda}(U), F^{-1}_{X_2|\Lambda}(U)) \stackrel{d}{=} (\Lambda, 1 - \Lambda),$$

and inequality (6) reads

$$\psi(X_1, X_2) \leq_{sl} \psi(\Lambda, 1 - \Lambda) \leq_{sl} \psi(\Lambda, \Lambda).$$

Henceforth, I denote by M_Λ the (or, in case of non-uniqueness, an) n-copula for which $M_\Lambda(F_{X_1}(x_1), \dots, F_{X_n}(x_n))$ is the joint distribution function of the random vector $(F^{-1}_{X_1|\Lambda}(U), \dots, F^{-1}_{X_n|\Lambda}(U))$. The fact that M_Λ can be different from M is elaborated on in Example 4.1.

4. Main results

This section extends Lemma 2.1 to the case of partial information on the dependence structure in the sense formalized in the previous section. Given $F_{X_1}, \dots, F_{X_n}, F_\Lambda$, and $F_{X_1|\Lambda}, \dots, F_{X_1|\Lambda}$, with $\int_{\text{supp}\Lambda} F_{X_i|\Lambda=\lambda}(x) \mathrm{d}F_\Lambda(\lambda) = F_{X_i}(x)$, $i = 1, \dots, n$, $x \in \mathbb{R}$, I define the family $\mathcal{C}^n_\Lambda$ of n-copulas as follows:

$$\mathcal{C}^n_\Lambda := \Big\{ C \in \mathcal{C}_n \mid C(F_{X_1}(x_1), \dots, F_{X_n}(x_n)) = \int_{\text{supp}\Lambda} C_\lambda(F_{X_1|\Lambda=\lambda}(x_1), \dots, F_{X_n|\Lambda=\lambda}(x_n)) \mathrm{d}F_\Lambda(\lambda) \text{ for every } \underline{x} \in \mathbb{R}^n \text{ for some } C_\lambda \in \mathcal{C}^n, \lambda \in \text{supp}\Lambda \Big\}. \quad (8)$$

Some remarks:

Remark 4.1. The definition of the class of copulas $\mathcal{C}^n_\Lambda$ is necessary since the choice of the particular set of conditional marginals $F_{X_i|\Lambda=\lambda}$, $\lambda \in \text{supp}\Lambda$, $i = 1, \dots, n$, may run out some dependence structures from $\mathcal{C}^n$. In this context, it is important to emphasize that the integral which defines $\mathcal{C}^n_\Lambda$ does **not** identify a so-called *convex sum* of copulas because the conditional marginals $F_{X_i|\Lambda=\lambda}$ in the integral are allowed to vary with λ.

Remark 4.2. Given that $F_{X_i|\Lambda=\lambda}(x)$, $i = 1, \dots, n$, is a d.f. for each $\lambda \in \text{supp}\Lambda$,

$$C_\lambda(F_{X_1|\Lambda=\lambda}(x_1), \dots, F_{X_n|\Lambda=\lambda}(x_n)),$$

with $C_\lambda \in \mathcal{C}^n$, is a joint d.f. for each $\lambda \in \text{supp}\Lambda$. Furthermore, as mixtures of d.f.'s are again d.f.'s,

$$\int_{\text{supp}\Lambda} C_\lambda(F_{X_1|\Lambda=\lambda}(x_1), \dots, F_{X_n|\Lambda=\lambda}(x_n)) \mathrm{d}F_\Lambda(\lambda)$$

is a d.f. as well, the marginals of which are $\int_{\text{supp}\Lambda} F_{X_i|\Lambda=\lambda}(x) \mathrm{d}F_\Lambda(\lambda) = F_{X_i}(x)$, $i = 1, \dots, n$. Now, Sklar's Theorem (see e.g., Nelsen (1999)) implies that $\mathcal{C}^n_\Lambda$ is not empty.

To illustrate Remark 4.1, consider the following example:

Example 4.1 (*Continuation of* Example 3.3). Under the assumptions of Example 3.3, $M \notin \mathcal{C}^2_\Lambda$, since

$$C_\lambda(F_{X_1|\Lambda=\lambda}(x_1), F_{X_2|\Lambda=\lambda}(x_2)) = \begin{cases} 1, & 1 - x_2 \leq \lambda \leq x_1; \\ 0, & \text{elsewhere}; \end{cases}$$

for any $C_\lambda \in \mathcal{C}^2$ so that in this case $\mathcal{C}^2_\Lambda$ consists only of the Fréchet lower bound:

$$\int_{\text{supp}\Lambda} C_\lambda(F_{X_1|\Lambda=\lambda}(x_1), \dots, F_{X_n|\Lambda=\lambda}(x_n)) \mathrm{d}F_\Lambda(\lambda) = \max\{x_1 + x_2 - 1; 0\}.$$

For a given copula C, I introduce the loss function

$$e_{C,\psi,\Lambda}(s) := \mathbb{P}[\psi(\underline{X}^C) < s] - \inf_{\tilde{C} \in \mathcal{C}^n_\Lambda} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}. \quad (9)$$

Clearly, since $\mathcal{C}^n_\Lambda \subset \mathcal{C}^n$,

$$\inf_{\tilde{C} \in \mathcal{C}^n_\Lambda} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\} \geq \inf_{\tilde{C} \in \mathcal{C}^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}.$$

I consider the following optimization problem:

$$\inf_{C \in \mathcal{C}^n_\Lambda} \left\{ \int_d^{+\infty} e_{C,\psi,\Lambda}(s) \mathrm{d}s \right\}. \quad (10)$$

Then, I state the main theorem:

Theorem 4.1. *For every real number d and every non-decreasing supermodular function ψ satisfying $\mathbb{E}[\psi(\underline{X}^{M_\Lambda})] < +\infty$, M_Λ is a minimizer of (10).*

Proof. Following the proof of Theorem 15 of Embrechts et al. (2005), I write:

$$\inf_{C\in\mathfrak{C}^n_\Lambda}\left\{\int_d^{+\infty} e_{C,\psi,\Lambda}(s)\mathrm{d}s\right\}$$
$$=\inf_{C\in\mathfrak{C}^n_\Lambda}\left\{\int_d^{+\infty}\left(\mathbb{P}[\psi(\underline{X}^C)<s]-\inf_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})<s]\right\}\right)\mathrm{d}s\right\}$$
$$=-\sup_{C\in\mathfrak{C}^n_\Lambda}\left\{\int_d^{+\infty}\left(\mathbb{P}[\psi(\underline{X}^C)\geq s]-\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\right)\mathrm{d}s\right\}$$
$$=\int_d^{+\infty}\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\mathrm{d}s-\sup_{C\in\mathfrak{C}^n_\Lambda}\left\{\int_d^{+\infty}\mathbb{P}[\psi(\underline{X}^C)\geq s]\mathrm{d}s\right\}$$
$$=\int_d^{+\infty}\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\mathrm{d}s-\sup_{C\in\mathfrak{C}^n_\Lambda}\left\{\int_d^{+\infty}\mathbb{P}[\psi(\underline{X}^C)> s]\mathrm{d}s\right\}$$
$$=\int_d^{+\infty}\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\mathrm{d}s-\sup_{C\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{E}[\left(\psi(\underline{X}^C)-d\right)_+]\right\}.$$

I distinguish between two cases: (i) assume

$$\int_d^{+\infty}\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\mathrm{d}s<+\infty.$$

Since $(\psi(\underline{x})-d)_+$ is supermodular, recalling Theorem 3.1 presented above,

$$\mathbb{E}[(\psi(\underline{X}^C)-d)_+]\leq\mathbb{E}[(\psi(\underline{X}^{M_\Lambda})-d)_+].$$

Furthermore, notice that $M_\Lambda\in\mathfrak{C}^n_\Lambda$ and $\mathbb{E}[\psi(\underline{X}^{M_\Lambda})]<+\infty$, or equivalently $\mathbb{E}[(\psi(\underline{X}^{M_\Lambda})-d)_+]<+\infty$ for all $d\in\mathbb{R}$, imply that

$$\sup_{C\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{E}[(\psi(\underline{X}^C)-d)_+]\right\}=\mathbb{E}[(\psi(\underline{X}^{M_\Lambda})-d)_+]<+\infty,$$

which proves the stated result. (ii) If

$$\int_d^{+\infty}\sup_{\tilde{C}\in\mathfrak{C}^n_\Lambda}\left\{\mathbb{P}[\psi(\underline{X}^{\tilde{C}})\geq s]\right\}\mathrm{d}s=+\infty,$$

then M_Λ is a trivial minimizer of (10). This completes the proof. □

Remark 4.3. M_Λ is best-possible in (10).

Example 4.2 (*Continuation of* Examples 3.1 *and* 3.2). Under the assumptions of Examples 3.1 and 3.2,

$$\mathrm{VaR}_\alpha\left[F_\Lambda^{-1}(V)\sum_{i=1}^n F_{Y_i}^{-1}(U)\right]=\int_0^1 F_\Lambda^{-1}(v)\mathrm{d}v\sum_{i=1}^n\mathrm{VaR}_\alpha[Y_i]=\mathbb{E}[\Lambda]\sum_{i=1}^n\mathrm{VaR}_\alpha[Y_i],$$

and

$$\mathrm{VaR}_\alpha\left[\sum_{i=1}^n F_{X_i}^{-1}(U)\right]=\sum_{i=1}^n\mathrm{VaR}_\alpha[X_i].$$

Let

- Λ follow a lognormal$(\mu_\Lambda,\sigma_\Lambda)$ law;
- Y_i follow a lognormal(μ_Y,σ_Y) law, $i=1,\ldots,n$;

so that X_i follows a lognormal(μ_X,σ_X) law with $\mu_X=\mu_\Lambda+\mu_Y$ and $\sigma_X=\sqrt{\sigma_\Lambda^2+\sigma_Y^2}$, $i=1,\ldots,n$. In that case

$$\mathrm{VaR}_\alpha\left[F_\Lambda^{-1}(V)\sum_{i=1}^n F_{Y_i}^{-1}(U)\right]=n\exp\left(\mu_X+\frac{1}{2}\sigma_\Lambda^2+\sigma_Y\Phi^{-1}(\alpha)\right),$$

while

$$\mathrm{VaR}_\alpha\left[\sum_{i=1}^n F_{X_i}^{-1}(U)\right]=n\exp\left(\mu_X+\sigma_X\Phi^{-1}(\alpha)\right).$$

One easily verifies that

$$\lim_{\alpha\to1}\frac{n\exp\left(\mu_X+\frac{1}{2}\sigma_\Lambda^2+\sigma_Y\Phi^{-1}(\alpha)\right)}{n\exp\left(\mu_X+\sigma_X\Phi^{-1}(\alpha)\right)}=0,$$

which confirms that in this case far in the tail of the loss distribution the VaR of $\sum_{i=1}^n F_{X_i|\Lambda}^{-1}(U)$ is dominated by the VaR of $\sum_{i=1}^n F_{X_i}^{-1}(U)$.

5. Conclusion

Consider a vector of risks of which the marginal distributions are known but the dependence structure is unknown. Suppose that there exists a common risk factor, with a given distribution function, conditionally upon which the marginal distributions of the vector of risks are known. Then there are two possible reasons to use the *improved comonotonic* or *conditionally comonotonic* dependence structure as introduced by Kaas et al. (2000):

1. As an approximation to the true dependence structure (see Dhaene et al. (2002a,b) and many subsequent papers), also when the true dependence structure is in effect known but the real d.f. under study is of a complicated form. For particular (light-tailed or moderately heavy-tailed[2]) cases encountered in practice this has proven to work well provided that the common risk factor is chosen (constructed) properly.
2. As a worst-case scenario since, as I prove in this note, it is the *most adverse* dependence structure in stop-loss and supermodular order and hence in Tail-VaR-based risk management [Theorem 3.1 and Corollary 3.1], and the *on-average-most-adverse* dependence structure in VaR-based risk management [Theorem 4.1]. Worst-case scenarios are not only very instructive under incomplete information, they are also crucial in stress-testing procedures. The recent credit crunch vividly illustrates the major importance of adequate stress tests for insurance and financial institutions.

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2 Laeven et al. (2005) show that in the presence of heavy tails, other approximation methods may be preferred.

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