Four Decades of Mathematical System Theory

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Abstract

As a follow-up to the volume *Three Decades of Mathematical System Theory* that the authors of this paper edited in 1989 at the occasion of the fiftieth birthday of Jan Willems, we discuss some trends that have become more visible in the past decade. It is argued that system theory is ready to play a role in many different fields of science. Three examples of nontraditional application areas are discussed.

1 Introduction

Ten years ago, the fiftieth birthday of Jan Willems was marked by the appearance of the volume *Three Decades of Mathematical System Theory. A Collection of Surveys at the Occasion of the 50th Birthday of Jan C. Willems* [18]. We, as editors of the volume, started the project in 1988. We decided that Jan could be best honored by collecting contributions from his closest collaborators (and friends) which together should form a survey of what had been achieved in system theory. Confronted with the problem of finding a title for the volume we quickly arrived at the *Three Decades* idea which seemed to have a nice ring to it; but then we had to prove that modern system theory was born in 1959. Fortunately we found the following quote from George S. Axelby, founding editor of both the *IEEE Transactions on Automatic Control* and the IFAC journal *Automatica*, which we repeat here for the benefit of the younger readers.[1]

> The year 1959 was the prelude to drastic changes in the control field (...). Then, the first IFAC Congress was held in Moscow, USSR in June 1960. Three papers were presented that were to revolutionize the theory of automatic control and set the direction of research for years to come. They were the papers by Kalman, Bellman, and Pontryagin. It seemed that almost immediately after the IFAC Congress

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[1] Quoted from a speech held in Los Angeles on December 10, 1987, at the 24th IEEE Conference on Decision and Control; *IEEE Control Systems Magazine* 8 (2), p. 98, 1988.

> all papers were involved in modern control theory and the use of state variables with the theorem, lemma, proof format.

The idea of editing a book with contributions from many of the most prominent researchers in the field was very well received. Almost all invitations we sent out were immediately accepted. By way of an extra bribe, at the end of 1988 we sent the prospective authors chocolate letters like most Dutch children (and grown-ups!) receive from Sinterklaas at the 5th of December. To which extent this helped is perhaps difficult to assess, but in any case we did manage to prepare the book manuscript in time — be it that the final version of one contribution had to take a shortcut directly from the authors, who will remain unnamed, to the printer's office.

Looking back ten years later, the first thing one notices when paging through the *Three Decades* volume is the wide range of text processors that have been used. The book might well serve as a showcase of word processing systems such as troff and ChiWriter which are now not more than dear memories. Developments in the past ten years have allowed the editors of the present volume to gather all contributions in one style, even though it is rumored that a few authors have thought it appropriate to submit their work in 2.09. Fortunately the contents of the papers in *Three Decades* are less sensitive to aging than the software used for their physical production, and many of the contributions are still excellent reading material.

The goal of this paper is not to review the texts from [18], although it would be an interesting exercise to see how certain directions have developed further. We also will not present an account of what happened in mathematical system theory in the past decade; rather we would like to take a look forward on the basis of a few trends. As marker points we shall take, together with the *Three Decades* volume, a SIAM report on "Future Directions in Control Theory" from 1988 [13], and a survey conducted for the *European Journal of Control* which was published in 1995 [6]. We will argue by means of a few selected cases that today's mathematical system theory is perhaps no longer focused on a single subject and has become a much more diverse field with beautiful and stimulating impulses from other sciences. This trend, which is to some extent also visible in mathematics as a whole, creates new challenges to the field of mathematical system theory. Perhaps the following quote from Jan Willems in [6] forms the best motivation for what follows:

> The art in control theory is to shape new questions, to introduce new concepts, to build new paradigms.

The examples to follow are chosen rather arbitrarily; in fact they are mainly chosen because of our own specific interests in these directions. It is obvious that others — including Jan Willems — would have opted for other themes. At the same time we believe that hybrid systems (section 2), control of chaos (section 3), and mathematics of finance (section 4) will play an important role in the next decade.

2 Hybrid systems

In the SIAM Report on Future Directions in Control Theory of 1988 [13], a separate section is devoted to *Discrete-Event and Hybrid Dynamical Systems*, as a subsection listed under *Other Research Opportunities*. On hybrid systems, the report notes the following:

> The control of hybrid dynamical systems will require a theory for dealing with a mixture of continuous and discrete variables with higher-level language symbols. Appropriate symbolic representation methodologies for such systems must be investigated in order to develop concepts that will enable us to model systems including both present state-of-the-art controllers and future intelligent control supervisory devices. The latter should be able to deal with decisions about the choice of a suitable control objective and strategies in the face of changes in the configuration or control environment, and would employ high-level linguistics instruction rather than conventional control signals.

In the volume *Three Decades of Mathematical System Theory* of 1989, the contribution by Roger Brockett [4] addresses systems which in some way combine discrete and continuous variables. Brockett discusses the use of continuous dynamical systems for carrying out discrete calculations, and more generally the simulation of finite automata by smooth dynamical systems. There is a relation to the supervisory control point of view expressed in the SIAM report, since in both cases the aim is to make a continuous system carry out certain tasks in response to discrete outputs. Some time later, in the opening article of the European Journal of Control [6] (1995), hybrid systems take a more prominent position. Hybrid systems are mentioned among the major open problems in systems and control by several respondents including Lennart Ljung, Peter Caines, and Pravin Varaiya. The thrust of the thinking on the subject can be seen from Vidyasagar's remark:

> Another interesting question is: 'How can one combine differential/difference equations with logical switches so as to enhance performance?' In some sense, this is the central question of intelligent control.

It seems therefore that by the mid-nineties hybrid systems have been clearly identified as a major new research area for systems and control theory.

Also during the nineties, continuous dynamics has gradually moved into the domain of attention of computer scientists. This holds in particular for those computer scientists who are involved in model checking and verification, a branch of computer science that is involved with the development of automated tools for the verification of the correctness of programs and hardware components. Interest in this area has moved from purely discrete systems to timed systems (in which real time plays a role), and from there to systems involving continuous dynamics; see for instance [17]. To allow the application of similar techniques as in the purely discrete case the continuous dynamics has to be very limited indeed. Considerable attention has been paid to piecewise constant vector fields and rectangular differential inclusions. One may argue that any smooth dynamical system may be approximated arbitrarily closely by a piecewise constant system, but of course there is a price to pay in terms of complexity. The tradeoff between complexity and accuracy offered by piecewise constant systems has in practical tests shown to be a rather difficult one [16, 15]. Perhaps it is too much to expect that formal methods could be of much help in the study of systems that incorporate substantial continuous dynamics, and one has to fall back on analysis of particular subclasses much in the way this is done in nonlinear control theory. However, concepts that have been studied in computer science such as modularity and hierarchy are of great importance in the modeling of complex systems.

To describe hybrid systems as a field of interaction between control theory and computer science would do injustice to the many scientists that in the past have contributed to the study of the interaction between discrete and continuous dynamics. In a sense the notion of different discrete states of a continuous system already comes up in the study of unilaterally constrained mechanical systems that was initiated by Fourier; here, the different discrete states (or "regimes") correspond to different sets of constraints being active or not. Fourier's results were extended by Farkas and have found widespread application in mathematical programming; a historical survey can be found in [21]. Fourier and Farkas were interested in equilibria so that no explicit dynamics appeared in their work. Also most of the applications in mathematical programming have been concerned with static systems. Recently however a program has been initiated by Dupuis and Nagurney [9] to study what they call "projected dynamical systems", which are dynamical systems that will switch regimes when certain constraints become active. Projected dynamical systems can be looked at as a systematic way of placing equilibrium analysis in a context of differential equations. There have been many other places in science and engineering where discrete and continuous aspects have been mixed — but we shall not enter into a discussion of quantum mechanics here.

Regime-switching behavior is not at all new to control theory. Some of the oldest references in optimal control theory are concerned with bang-bang control, and so the study of differential equations with discontinuous right-hand sides was recognized early as a natural topic in optimal control [5]. Ideal relays, which are regime-switching elements, have been studied for a long time as part of control schemes; see for instance Tsypkin's book [23]. This work in itself can be placed in a tradition of analysis of discontinuous dynamical systems which started in the forties in the Soviet Union, mainly under the impulse of Andronov [1]. A standard reference on differential equations with discontinuous right-hand sides is of course the book by Filippov [12]. In the context of control design one may also refer to Utkin's work on sliding mode control [24]. In nonlinear control theory it is a well-established fact that nonsmooth control is intrinsically more powerful than smooth control [3] and especially in recent years considerable effort has gone into the design of switching control schemes. In industry, switching controllers (for instance programmable logic controllers) are in fact often the rule rather than the exception. The motivation behind the use of switching control in industrial applications is usually not so much increased power, but rather simplicity which aids both implementation and understanding by a human operator. It might be worthwile to study several proposals made under the heading “fuzzy control” from a point of view of switching control.

In the world of simulation languages the mixing of discrete and continuous dynamics has been a topic of interest for a long time, and activity in this area has in fact been increasing recently. A very important way of keeping the complexity of a model description under control is to choose a time scale of interest and to refrain from detailed modeling of anything that happens on a different time scale. This means that things that change on a slower time scale will be treated as constants, and things that change on a faster time scale will be modeled as happening instantaneously. For instance, in the description of the motion of a bouncing ball it would be quite acceptable in many cases to model the impacts as instantaneous; the description of a decompression and a decompression phase of the ball would in fact be arduous and one would prefer to avoid it if it is not strictly needed. As a consequence, one will have to accept a certain discontinuity in the model. Simulation languages such as ACSL have offered functionality for incorporating such discontinuities from the very beginning. As an example of a more recent effort, the popular simulation tool Simulink that goes with the Matlab environment has been expanded with a flowgraph utility that is intended to allow the user to keep track of discrete state changes. In the design of the object-oriented physical modeling language Modelica [10] there has been special attention towards proper handling of mode switches.

Representation, analysis, and design of hybrid systems are all areas which are in an early stage of development. As the area covered by the term “hybrid systems” is very large indeed, it is likely that no uniform approach will be successful for all applications and that different methodologies will be developed in different subareas, using additional structure that is available in these environments. Concerning representation, automata-based models (not surprisingly mainly proposed by computer scientists) have been popular; equation-based models would be closer to the standard paradigm of control theory and are in principle just as amenable to parallel composition, the key to combating complexity. Hybrid systems present a challenge to behavioral theory as developed by Jan Willems and coworkers, because the time axis is punctuated by events in a way that is in general different for different trajectories of the same system, in contrast to the assumption in standard behavioral theory that the time axis is the same for all trajectories. It is expected that such difficulties can be overcome, though. More important is the emphasis placed by Willems on the modeling of complex systems by interconnecting models of subsystems, and the role of system theory in this. It is precisely this notion of “interconnection” or “parallel composition” which is likely to play a crucial role in the study of complex hybrid systems.

We refer the reader to the forthcoming book [20] for a much more extensive discussion of hybrid systems. In dealing with this subject area, systems theorists must have an outward orientation. They should look to computer science for ways of handling complexity and for motivating questions such as safety, in addition to the traditional performance criteria of systems and control theory like stability. But they should also look to operations research for extensive knowledge about systems of inequalities, to applied mechanics and circuit simulation for the construction of large models incorporating mode switches, and to industrial control for tried and tested methods of designing control schemes based on switching. We believe that modern systems and control theory offers an excellent background to do all of this.

3 Control of chaos

In the 1988 SIAM Report [13], *Control of Chaos* is listed as one of the "other research opportunities" that is, one of the

> new opportunities in which the connections to the existing theory are not firmly established. However, the importance of these topics indicate that the appropriate mathematical developments of new control paradigms should occur.

Looking back at the recent history of the control of chaos field, it is worth noting that the report [13] mentions among speculative ideas (!) two interesting directions, namely control in the vicinity of bifurcation points, and steering towards a chaotic regime. Indeed, the last decade has shown a tremendous activity in the control of chaos field; see e. g. the bibliography of Chen [7] or the extensive list of references in [8], with over 700 references dealing with various aspects of chaos control. However, a large majority of these papers are not within the traditional control literature but are mostly published in different areas such as physics, dynamical systems theory, and circuit and communication theory. Although there have been some very clear ideas of exploiting the control of bifurcations perspective, with some promising applications, see e.g. [22], most of the chaos control literature identifies the paper [19] as a breakthrough in the field. In this paper, which appeared in the Physical Review Letters, Ott, Grebogi, and Yorke developed what is later often referred to as the OGY method. The interested reader is referred to the paper, or the more recent account [14], for a detailed exposition of the OGY method.

Basically, the method can be understood as follows. Given a nonlinear dynamics that for some nominal parameter value exhibits chaotic (complex) dynamics, the goal is to direct the system trajectories towards some periodic orbit inside the chaotic attractor. The OGY method then suggests to linearize the nominal dynamics about the desired trajectory and to use a simple linear controller once the trajectories get sufficiently close to the desired trajectory — which eventually will happen since one is dealing with a goal trajectory belonging to a (chaotic) attractor. Besides some subtle shortcomings of the methodology (e. g. the time dependent nature of the linearization, and the local validity of the feedback) further improvements, variants, and perhaps most importantly, numerous applications of the OGY method have been reported. See [8] for a recent account on this.

Although some of the applications foreseen in the physics and communications literature are for the moment mainly of a speculative character there are breathtaking attempts in biomedical engineering of using the OGY method. One example of this type deals with tackling cardiac arrhythmia. The basic idea in this case is that in normal circumstances, the amount of intracellular Ca is cyclically increased and decreased as in a system of coupled oscillators, and may resemble some kind of chaotic behavior. The cyclical changes in the amount of intercellular Ca are caused by the heart (arrhythmogenic) mechanism. There are emerging ideas to control (with some advanced pacemaker) the chaotic arrhythmia. Another, earlier case where chaos — or better complex dynamics — is used for control in reality, concerns the steering with little energy of a spacecraft, using geostationary orbits of earth and moon, to encounter a comet after a millions of miles long travel; see [11]. The reader will notice that this work, performed by NASA using extensive computer resources, preceded the appearance of the OGY paper and was even earlier than the SIAM report.

As mentioned before, most of the control of chaos literature is at the moment outside the standard systems and control literature. In fact, one could argue that many of the chaos control papers that have appeared in the physics literature would have been received very critically in our field. In a sense, in many of these papers, the emphasis is more on ideas and illustrations — both simulations and experiments — than on advanced control. Nevertheless these papers deal with (systems and) control, and the systems and control community so far has hardly contributed in this direction. Despite the timely impulse of the report [13] the control community appears to have largely neglected the chaos control area. This is the more remarkable since in control of chaos one tries to exploit characteristics of the underlying dynamics, an idea which is prominently present in today's systems and control theory. Apparently, chaos so far has been too distinct — and difficult — for most of the researchers in our field! On the other hand, it is clear that still much remains to be done in the wide area of chaos control. One may very well argue that systems and control provides an excellent basis for studying many of challenging topics in chaos control. In this regard one can consider, see for instance [8], (controlled) synchronization, delayed feedback control, chaos control in distributed systems, chaos synchronization based communication and so on. All these subjects contain a wealth of difficult and challenging questions to be solved, by us!

4 Mathematics of Finance

There is a brief remark in the SIAM report [13] which involves applications in the area of finance. One of the paragraphs in the section on stochastic control reads:

> Problems of dynamic decision-making under uncertainty in management science and economics continue to provide interesting re-

> search topics in stochastic control. The popularity of the Black-Scholes model for pricing of stock options has led to much interest in stochastic control and optimal stopping models in financial economics. Other areas of application include hydrology and fishery management.

The *Three Decades* volume has a contribution by Manfred Deistler on identification with applications to econometrics, but no specific connection is made to finance. In the survey conducted for EJC, none of the respondents (not even Mark Davis) mentions mathematical finance as a source for open problems or challenges. Apparently the subject was considered to be too far away from traditional control.

Nevertheless, it is possible to look at the theory of pricing and hedging of financial derivatives as control theory with a twist. The observation that portfolio composition may be looked at as a stochastic optimal control problem is certainly not new; in fact the application of dynamic programming methods to portfolio problems is among the classical contributions of Robert C. Merton to modern finance. However, the key point of the Black-Scholes theory of option pricing is not so much optimization, but rather what might be called *equalization*. As in the optimal control case, there is a cost functional defined on trajectories of a certain dynamical system and there is a control function, but the underlying dynamical system is indeterministic (there are many state trajectories even when the initial condition and the control strategy are fixed) and the purpose of control is not to optimize the cost but rather to eliminate the variation of the cost over different trajectories. The constant cost, if it can be achieved, gives the price of the option.

The setting described above is certainly different from what is usually considered in control theory. Also the trajectories that are being used may look rather unfamiliar. It is in fact essential that the theory is based on highly irregular state trajectories in order to produce results that are more or less in accordance with the observed reality. The standard setting used in modern financial theory makes use of stochastic differential equations, but it has been shown that this is not strictly necessary [2]. One may start from a collection of deterministic trajectories (a behavior, one might say), but to get interesting results these need to be nowhere differentiable functions and so the corresponding behaviors are not likely candidates for description by ordinary differential equations.

In spite of these substantial differences, there are also some interesting parallels between modern finance and control theory. As modern financial theory is maturing these parallels actually become more visible. Traditionally much emphasis in finance has been placed on the *pricing* of derivative products. As it happens, the pricing problem may be solved without explicit consideration of any control strategy by introducing a suitable measure on the collection of possible trajectories of the underlying asset; the price can be computed as an expected value with respect to this measure. One can then derive a corresponding control strategy (the so-called "delta hedge") from the price as a function of the time and of the price of the underlying asset. The delta strategy theoretically achieves the stated control purpose of equal costs along all trajectories.

There are some practical problems involved in the implementation of the delta strategy, arising from transaction costs and possible lack of liquidity. More fundamentally however, the standard theory is based purely on some given nominal model which is taken to represent reality exactly. To a control theorist this issue is of course all too familiar. Financial risk managers are also well aware of what in the business is often called "model risk". Whereas the theory of pricing of financial products has reached a high degree of sophistication, the theory of robust hedging is in an early stage of development. Control theorists may well take their chances here. In addition, there are a number of other issues in finance which might be fruitfully attacked by system theorists, for instance relating to filtering and identification.

5 Conclusions

In four decades, systems and control theory has matured into a well-defined mathematical discipline representing a large body of knowledge. Looking into the future, it seems that the field will benefit from a more outward orientation which will increase its visibility. By its nature as a generic discipline, systems theory has the potential of playing a role in many areas of science and engineering. However, this potential will not be used automatically; systems theory has to prove its value in every single case anew, and for this the specific nature of each particular application area has duely to be taken into account.

In this paper, we have briefly discussed three research areas where we feel that systems and control theory has a contribution to make. We believe that the topics presented here are representative of the wide range of possible avenues to explore. Systems theorists can feel safe in entering all these separate fields knowing that a solid foundation for their discipline has been laid, not in the last place by the man to whom this volume is devoted: Jan Willems.

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