Chapter 18. Negation

Heinrich Wansing

Subject
Logic and Language » Logic
DOI:
10.1111/b.9780631206934.2001.00021.x

18.1. Introduction

This chapter is concerned with logical aspects of negation, i.e. with the role of negation in valid inferences and hence with the contribution negation makes to the truth and falsity conditions of declarative expressions. Negation is an important philosophical and logical concept. Often differences between logical systems can – at least partially – be described as differences between the notions of negation used in these logics. In natural deduction, for example, classical logic can be obtained from intuitionistic logic on the addition of the double negation elimination rule:

$$\neg\neg A / A$$

Notwithstanding the importance of negation, the immense literature on negation[1] abounds with disagreement. According to Gabbay (1988), intuitionistic negation is a typical instance of negation as inconsistency, while according to Avron (1999), intuitionistic negation clearly fails to be a genuine negation. In the opinion of Tennant (1999), basing negation on the notion of disproof leads to negation in intuitionistic relevant logic, whereas by treating the notion of disproof on a par with the notion of proof, López-Escobar (1972) obtains the strong, constructive negation of Nelson (Almukdad and Nelson, 1984; Nelson, 1949; 1959), and independently investigated by von Kutschera (1969). Moreover, it can be shown that the latter negation fails to be a negation as inconsistency in the sense of Gabbay.

The question arises: What are at least necessary conditions under which a unary connective ought to be regarded as a negation operation? The disagreement about negation is, however, even more fundamental. According to Zwarts (1996), for instance, linguistic research has clearly revealed that negation in natural languages occurs in various syntactic categories, including sentence negation. However, Englebretsen (1981), for example, argues to the effect that whereas one ought to draw the Aristotelian distinction between predicate and predicate term negation, there is no such thing as external, sentential negation. Thus there is even disagreement concerning the syntactic types to which negation belongs.

Various strategies are available to obtain a single, uniform account of the multiplicity of syntactic types of negation or to reduce in the formal analysis the number of syntactic types in which negation occurs in natural languages:

- Elimination: to explain away certain syntactic types
- Generalization: to give an account such that various syntactic types of negation emerge as special cases of a general construction
- Representation: to represent one type of negation in terms of another type

In section 18.2 instantiations of the first two strategies will be dealt with. Section 18.3 is then devoted to representing negation by means of unary connectives with a special emphasis on motivating strong, constructive negation. Moreover, various approaches toward defining and classifying notions of sentential negation will be surveyed. Some summarizing general ideas are assembled in section 18.4.

18.2. The Syntactic Categories of Negation

In the literature on syntactic categories in natural languages, it has sometimes been suggested to treat the Boolean particles 'not', 'and', and 'or' as variably polymorphic operations. In the case of 'not', this means that for any syntactic type (including sentences), 'not' may be combined with an expression of that type to form a compound expression of the same type; see, for example, van Benthem (1991, pp. 26f. and ch. 13). However, there is no general consensus on this variable polymorphism. The proponents of a neo-Aristotelian term logic, for example, have called into question the nowadays orthodox view that negation is a unary sentential connective. Most explicitly, Englebretsen (1981) has tried to explain away sentential negation, and the present section is, among other things, concerned with a critical examination of this eliminative view.

18.2.1. The neo-Aristotelian elimination of sentence negation

According to the Aristotelian term logic (as presented, for example, in Englebretsen (1981), Horn (1989, ch.1), and Sommers (1982), every sentence consists of exactly one subject and exactly one predicate. Both the subject and the predicate are possibly complex terms. In the sentence

John is pleased.

the expression 'John' is the subject, the expression 'is pleased' is the predicate, and the expression 'pleased' is the predicate term. As Sommers (1982, p. 287) explains, "There is no reason to factor the predicate into a part that is the predicate term and a part that is the copula" 'is'. If "the terms are not explicit, the traditional logician will regiment the proposition to bring out its logical form. Thus 'Socrates runs' could be regimented as 'Socrates is a runner'." Every sentence affirms or denies what is denoted by its predicate of what is denoted by its subject. While

John is pleased.

affirms pleased of John.

John is not pleased.

denies pleased of John. This form of negation is called predicate negation (or predicate denial) and is to be distinguished from predicate term negation. In the case of predicate term negation, a predicate term is negated to obtain another predicate term. The predicate term negation of ‘pleased’ for instance is ‘not-pleased’, and the sentence

John is not-pleased.

affirms the predicate ‘not-pleased’ of John. If the predicate term of a sentence is negated, this results in a contrary of that sentence. A pair of contrary sentences cannot both be true. Whereas a predicate term ‘P’ may have many contraries, according to the neo-Aristotelian term logicians, it has exactly one logical contrary, namely ‘not-P’ (or ‘non-P’). Among the non-logical contraries of the predicate term ‘ancient’, for example, are ‘medieval’ and ‘modern’. If the predicate of a sentence is negated, one obtains a contradictory of that sentence. A pair of contradictory sentences can neither both be false nor both be true. Whereas the predicate term negation of a sentence implies the predicate negation of that sentence, the converse is not true. In this sense, predicate term negation is stronger than predicate denial.

Actually, the distinction between logical and non-logical contraries is quite subtle. There are passages in Aristotle's writings suggesting that contrariety is a polar notion (Horn, 1989, p. 37ff.) presupposing two extreme points of a scale:

Since things which differ from one another may do so to a greater or a less degree, there exists also a greatest difference, and this I call ‘contrariety’.

(Aristotle, Metaphysics 1055a 17–28)

Horn (1989, p. 39) distinguishes between

(i) contrariety simpliciter

(ii) immediate (alias strong or logical) contrariety, and

(iii) mediate (alias weak or non-logical) contrariety.

If by the span of a predicate term ‘P’ one means the class of entities that can be either P or not-P, two predicate terms are contraries simpliciter of each other iff (if and only if) both have the same span and what they denote cannot both be true of any element from their span. If two contrary predicate terms ‘P’ and ‘Q’ are such that for any a from their shared span, the sentences ‘a is P’ and ‘a is Q’ form a pair of contradictory sentences, the terms are said to be immediate contraries. This is the case if the scale associated with ‘P’ and ‘Q’ is binary. As Sommers (1982, p. 168) puts it, “[a] pair of logical contraries exhausts a range of predicability.” When affirmed of numbers, ‘even’ and ‘odd’ form a pair of immediate contraries. Mediate contraries are contraries that are not immediate. If two mediate contraries denote the extremes of a (more than binary) scale, they are said to be polar, and otherwise they are called simple. While, for instance, ‘black’ and ‘white’ are lexicalized polar contraries, ‘black’ and ‘red’ are simple contraries. Any natural language predicate term has at most one polar contrary (with respect to a given scale). Also the immediate contrary of a given term is unique if it exists.[2] It would also make sense to classify immediate contraries as polar, if being polar is understood in the more general sense of forming the extremes of an at least binary scale. Both immediate and mediate non-simple contraries would then be polar in this more general sense.

The situation is more complicated if categorically mistaken sentences are also taken into account. Whereas ‘John is well’ and ‘John is ill’ are contradictories because ‘well’ and ‘ill’ are immediate contraries (at least according to Aristotle and Horn), ‘2 is well’ and ‘2 is ill’ fail to be contradictories, since neither of these categorially mistaken sentences is true. Likewise, if the name ‘John’ is non-denoting, then neither ‘John is well’ nor ‘John is ill’ are true.[3]

The term logical contrary suggests a correlation with some systematic syntactic device that – at the level of formalization – goes beyond merely pairing representations of lexical items like ‘even’ and ‘odd.’ Englebretsen (1981), Sommers (1982), and Horn (1989) use the predicate term negation ‘not-P’ (‘non-P’) to form the immediate (alias logical) contrary of a predicate term ‘P’. The question arises how this is related to the morphology and the lexical repertoire of natural languages. Predicate term forming prefixes like ‘dis-’, ‘un-’, and ‘im-’ do not in general generate immediate contraries but often map a predicate term to its polar non-immediate contrary.[4] A person needs, for instance, neither to be pleased with a certain situation nor to be displeased with that situation. She might just not care. Nor need a person be either happy or unhappy. According to Englebretsen (1981, p. 50), category mistakes are the only sources of violations of the Law of Bivalence: “Every sentence which is sensible, category correct, is true or false.” But as has just been seen, predicate term negation may also violate the Law of Bivalence in categorically correct sentences.

Since general polar contrariety is a more general concept than immediate contrariety insofar as polar contraries are immediate if the associated scale is binary, and since, moreover, the predicate term forming morphemes ‘dis-’, ‘un-’, and ‘im-’ often yield mediate non-simple contraries, it seems justified to use ‘not-P’ (‘non-P’) to form the unique polar contrary of a predicate term ‘P’ (in the sense of Horn) and not its immediate contrary. A further justification for this convention is that, in the case of categorically correct sentences, there is no need to distinguish between denying a predicate and affirming its immediate contrary, so that, for categorically correct declarative sentences, forming predicate term negation in a canonical way may be seen to yield unique polar contraries.

In an attempt to explain away sentence negation, as a first step, Englebretsen (1981, p. 36) attacks the view that every negation is sentential:

Modern sentential logicians … fail to recognize anything but sentential negation. For such logicians any negated term must somehow be rendered part of a negated sentence. Nevertheless, notwithstanding what might look like notational economy, there are some sentences which cannot be analyzed in such a logic – sentences which are true and make use of term negation, but which become false or senseless when term negation is replaced by sentential negation. An example of such a sentence is ‘Numbers are neither coloured nor noncoloured (uncoloured)’. Note that what is noncoloured is colourless, transparent, invisible, etc. Helium is noncoloured but not the number 2. The mathematical logician reparses the first as ‘All things which are numbers are neither coloured nor noncoloured’; then as ‘Everything is such that if it is a number then it is not coloured and not noncoloured’, then as ‘Everything is such that if it is a number then it is not the case that it is coloured and it is not the case that it is not the case that it is coloured’, and finally as ‘$\forall x(Nx \supset (\neg Cx \wedge \neg\neg Cx))$’. But in the usual first order predicate calculus this entails that nothing is a number! [notation adjusted]

However, this argument clearly fails to show that there are sentences which cannot be represented using sentential negation. Instead, it demonstrates that using a single sentential negation operation to represent both predicate negation and predicate term negation may have undesirable consequences. Using the unary connective ‘¬’ to represent predicate negation and the unary connective ‘~’ (strong negation) to represent predicate term negation, the translation of Englebretsen's example into predicate logic is:

$$\forall x(Nx \supset (\neg Cx \wedge \neg{\sim} Cx))$$

‘Everything is such that if it is a number then it is not the case that it is coloured and it is not the case that it is non-coloured’.
In a second step, Englebretsen intends to show that “what is negated is never a sentence” (1981, p. 45):
[W]hen ‘p’ is ‘S is P’, then ‘It is not the case that p′ (‘¬p’) is ‘S is not P’. Thus, ‘It is not the case that 9 is even’ is ‘9 is not even’. But sometimes ‘it is not the case that’ can be read as ‘it is untrue that’. For example, ‘It is not the case that I have stopped beating my wife’ does not mean ‘I have not stopped beating my wife’, but rather ‘It is untrue that I have stopped beating my wife’. In other words ‘it is not the case that’ is ambiguous. Usually it is the ‘not’ of predicate denial. Sometimes it is the predicate ‘untrue’. The mathematical logician always takes it in the second way since he does not recognize predicate denial.
(1981, p. 47f.) [notation adjusted]
He then continues to explain that “[i]f ‘it is not the case that’ is usually a sign of denial and sometimes a metalinguistic predicate (viz. ‘untrue’), then it seems we are well on our way to the position that no negation is sentential” (1981, p. 49). First, one might object that
It is not the case that I have stopped beating my wife.
is itself ambiguous and may well mean that I have not stopped beating my wife. Second, if the external ‘it is not the case that’ may be read either as predicate negation or as a meta-linguistic predicate that can be affirmed or denied of propositions, one might also draw the conclusion that predicate denial and this meta-linguistic predicate are to be understood as external, sentential negation.
This is not the place for a general discussion of Aristotelian and neo-Aristotelian term logic. What the term logicians correctly point out is that a distinction must be drawn between predicate negation and predicate term negation. A natural idea is to represent these forms of negation by distinct unary connectives. Obviously, such a representation abstracts away from the innersentential syntactic realization of predicate negation and predicate term negation in natural languages. The representing connectives can be iterated and therefore, for instance, be interpreted as algebraic operations. To formally take into account the distinction between predicate negation and predicate term negation, it is therefore interesting to investigate algebraic structures comprising algebraic counterparts of at least two sentential negations.

18.2.2. Generalization on the basis of 2-negation algebras

Yet there must be a way to link the internal operators on predicates with the external operator (operators) on propositions, a bridge between the logic of terms and the logic of propositions. (La Palme Reyes et al., 1994, p. 50)
La Palme Reyes et al. (1994, 1999) develop a category-theoretic model for a pair of negations that can be applied to both predicates and sentences, “thereby incorporating in a single context both term logic and propositional logic” (1994, p. 51). This instance of the generalizing approach is based on the notion of a 2-negation algebra. A 2-negation algebra is a bounded distributive lattice with two unary operations (¬, negation, and −, supplement):

$$(\mathbf{B}, \leq, \vee, \wedge, 0, 1, \neg, -)$$

While negation is required to satisfy

$$y \leq \neg x \text{ iff } y \wedge x = 0 \quad (18.1)$$

supplement must satisfy

$$-x \leq y \text{ iff } x \vee y = 1 \quad (18.2)$$

La Palme Reyes et al. then use ¬ to represent predicate term negation and use — to represent predicate negation.[5] This choice is surprising, because (18.1) defines the pseudocomplement of Heyting algebras. [See chapter 11.] The pseudo-complement is the algebraic counterpart of intuitionistic negation, which is a contradiction forming operation and not a contrary forming one.
The approach by La Palme Reyes et al. is not presented here in any detail, because it does not account for a certain important difference in terms of inference patterns between predicate negation and predicate term negation. Whereas predicate negation satisfies contraposition as a rule, predicate term negation in general does not. If the predicate term of a sentence is negated resulting in the polar contrary of that sentence but not in its contradictory, then contraposition fails. If from
John is pleased (in his work).
it can be derived that
Jack is pleased (in his work).
it may nevertheless be the case that
John is displeased (in his work).
cannot be derived from
Jack is displeased (in his work).
because it may be that Jack is displeased and John is neither pleased nor displeased. However, it is well-known that any pseudocomplement ¬ in a bounded lattice satisfies[6]

$$x \leq y \text{ iff } \neg y \leq \neg x \quad (18.3)$$

Contraposition for the supplement follows by duality. Therefore, although 2- negation algebras are interesting, natural and simple algebras, they do not seem to be the right kind of structures to represent predicate term negation.

18.3. Representing Negations as Sentential Operations

This section shows that predicate term negation can be represented as a sentential negation that has an independent motivation, namely as strong, constructive negation. Moreover, this section considers various suggestions for defining and classifying notions of sentential negation that have been made in the literature.

18.3.1. Negation as falsity

There are several distinct justificatory roads to negation in the sense of definite falsity; see, for instance, Pearce (1991). One of these roads is provided by Kripke frames interpreted in terms of information states (or pieces) partially ordered by a relation of ‘possible expansion of information states.’ The basic idea is that an information state may not only support the truth of certain atomic formulas but also support the falsity of certain atomic formulas; see Gurevich (1977), López-Escobar (1972), Routley (1974), and Thomason (1969). In other words, the idea is to treat verification and falsification on a par as equally important primitive semantic relations. These considerations give rise to the notion of a Nelson model. A

Nelson model is a structure $\langle \mathbf{I}, \sqsubseteq, v^+, v^- \rangle$, where $\langle \mathbf{I}, \sqsubseteq \rangle$ is a partially ordered set and both v^+ and v^- are valuation functions assigning to every propositional variable p a subset of **I**. Intuitively, v^+ sends atoms to the information states at which they are verified, whereas v^- sends atoms to the information states at which they are falsified. Moreover, it is required that for every propositional variable p and every t, u ∈ **I**:

$$\text{Persistence}^+ \quad \text{if } t \sqsubseteq u, \text{ then } t \in v^+(p) \text{ implies } u \in v^+(p)$$

$$\text{Persistence}^- \quad \text{if } t \sqsubseteq u, \text{ then } t \in v^-(p) \text{ implies } u \in v^-(p)$$

Let M= $\langle \mathbf{I}, \sqsubseteq, v^+, v^- \rangle$ be a Nelson model, t ∈ **I** and A a formula in the language with strong negation ~, intuitionistic implication $\supset_b$, conjunction ∧, and disjunction ∨ over the denumerable set Atom of propositional variables. The notions M, t $\models^+$ A (A is verified at t in M) and M, t $\models^-$ A (A is falsified at t in M) are inductively defined as follows:

$$\mathcal{M}, t \models^+ p \quad \text{iff} \quad t \in v^+(p),\ p \in Atom$$

$$\mathcal{M}, t \models^- p \quad \text{iff} \quad t \in v^-(p),\ p \in Atom$$

$$\mathcal{M}, t \models^+ B \wedge C \quad \text{iff} \quad \mathcal{M}, t \models^+ B \text{ and } \mathcal{M}, t \models^+ C$$

$$\mathcal{M}, t \models^- B \wedge C \quad \text{iff} \quad \mathcal{M}, t \models^- B \text{ or } \mathcal{M}, t \models^- C$$

$$\mathcal{M}, t \models^+ B \vee C \quad \text{iff} \quad \mathcal{M}, t \models^+ B \text{ or } \mathcal{M}, t \models^+ C$$

$$\mathcal{M}, t \models^- B \vee C \quad \text{iff} \quad \mathcal{M}, t \models^- B \text{ and } \mathcal{M}, t \models^- C$$

$$\mathcal{M}, t \models^+ B \supset_b C \quad \text{iff} \quad (\forall u \in \mathbf{I}) \text{ if } t \sqsubseteq u, \text{ then } \mathcal{M}, u \models^+ B \text{ implies } \mathcal{M}, u \models^+ C$$

$$\mathcal{M}, t \models^- B \supset_b C \quad \text{iff} \quad \mathcal{M}, t \models^+ B \text{ and } \mathcal{M}, t \models^- C$$

$$\mathcal{M}, t \models^+ \sim B \quad \text{iff} \quad \mathcal{M}, t \models^- B$$

$$\mathcal{M}, t \models^- \sim B \quad \text{iff} \quad \mathcal{M}, t \models^+ B$$

With this definition, verification and falsification of arbitrary formulas are persistent with respect to ⊑. Semantic consequence is defined as follows: $\Gamma \models_{N4}$ A iff for every Nelson model M= $\langle \mathbf{I}, \sqsubseteq, v^+, v^- \rangle$ and every t ∈ **I**, if M, t $\models^+$ B for every B ∈ Γ, then M, t $\models^+$ A. Nelson's propositional logic **N4** is the theory of the class of all Nelson models in the given language. **N4** conservatively extends positive propositional logic, the positive part of intuitionistic logic. [See chapter 11] **N4** can be axiomatized by adding to an axiomatization of positive logic the following axiom schemata:

$$A1 \quad \sim\sim A \equiv A$$

$$A2 \quad \sim (A \wedge B) \equiv (\sim A \vee \sim B)$$

$$A3 \quad \sim (A \vee B) \equiv (\sim A \wedge \sim B)$$

$$A4 \quad \sim (A \supset_b B) \equiv (A \wedge \sim B)$$

where A ≡ B is defined as $(A \supset_b B) \wedge (B \supset_b A)$.

Contraposition as a rule does not hold in **N4**.[7] Moreover, provable equivalence fails to be a congruence relation on the set of formulas. If formulas A and B are defined as being strongly equivalent iff both A, B and their strong negations ~A, ~B are provably interderivable, then it can be shown that strong equivalence is a congruence relation in **N4**, i.e., there is a replacement theorem for strongly equivalent formulas. **N4** is a system of paraconsistent logic, because not for every B, {A, ~A} $\vdash_{N4}$ B. Moreover, **N4** is a system of four-valued logic.[8] Every pair (V^+, v^-) of valuations induces a valuation v: **I** × Atom → {1, 0, Ø, {1, 0}} by defining:

$$v(t, p) = 1 \quad \text{iff } t \in v^+(p)$$

$$v(t, p) = 0 \quad \text{iff } t \in v^-(p)$$

$$v(t, p) = \{1, 0\} \quad \text{iff } (t \in v^+(p) \text{ and } t \in v^-(p))$$

$$v(t, p) = \varnothing \quad \text{iff } (t \notin v^+(p) \text{ and } t \notin v^-(p))$$

The model M= $\langle I, \sqsubseteq, v^+, v^- \rangle$ and the induced model M′ = $\langle \mathbf{I}, \sqsubseteq, v \rangle$ validate the same formulas of the language under consideration, if for every p ∈ Atom, we define

$$\mathcal{M}', t \models^+ p \quad \text{iff } v(t, p) = 1$$

$$\mathcal{M}', t \models^- p \quad \text{iff } v(t, p) = 0$$

The three-valued logic **N3** is the theory of the class of all Nelson models $\langle \mathbf{I}, \sqsubseteq, v^+, v^- \rangle$, where for every atom p, $v^+(p) \cap v^-(p)$ is empty. **N3** can be axiomatized by adding to an axiomatization of **N4** the ex contradictione schema $\sim A \supset_b (A \supset_b B)$. Strong negation in Nelson's logics **N3** and **N4** is also referred to as constructive negation, since in both systems negation satisfies

$$\textit{Constructible falsity} \quad \vdash \sim (A \wedge B) \text{ iff } (\vdash \sim A \text{ or } \vdash \sim B)$$

In **N3** the contradictory forming intuitionistic negation $\neg_h$ can be defined using the primitive, contrary forming strong negation ~ and intuitionistic implication ⊃h:

$$\neg_h A \text{ iff } A \supset_h \sim A$$

N3 provides a natural example of a logical system with two kinds of negation suitable for representing both predicate denial and predicate term negation.[9]

Considerations on the so-called Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operations in terms of direct (or canonical) proofs (alias constructions) [see chapter 11] may also lead to Nelson's systems. According to this interpretation, a canonical proof of ($A \supset_h B$), for instance, is a construction that applied to a proof of A results in a proof of B. López-Escobar (1972) suggested supplementing the BHK interpretation by the notion of (canonical) disproof. He gives the following disproof interpretation of the intuitionistic connectives ∧, ∨, and ⊃h and the strong negation ~ (notation adjusted):

(i) The construction c refutes A ∧ B iff c is of the form 〈i, d〉 with i either 0 or 1 and if i = 0, then d refutes A and if i = 1 then d refutes B.

(ii) The construction c refutes A ∨ B iff c is of the form 〈d, e〉 and d refutes A and e refutes B.

(iii) The construction c refutes $A \supset_b B$ iff c is of the form 〈d, e〉 and d proves A and e refutes B.

(iv) The construction c refutes ~A iff c proves A.

Under the proof and disproof interpretation, a proof of ~~A is not conceived of as a proof of A ⊃h ⊥ (‘A implies absurdity’), but rather as a refutation of A. This is a completely natural and direct way of relating proofs and disproofs by means of negation. Lopez-Escobar uses the following notion of provable sequent, with respect to which Nelson's logic **N4** emerges as sound:

$\{A_1, \ldots, A_n\} \rightarrow A$ is *valid* iff there exists a construction π such that $\pi(c_1, \ldots, c_n)$ proves A, whenever $c_1, \ldots, c_n$ are constructions proving $A_1, \ldots, A_n$ (if $1 \leq n$).

A sequent Ø → A is said to be valid iff a construction exists that proves A. Moreover, López-Escobar assumes that no construction both proves and disproves the same A. Note that {A, ~A} → B is valid under the stronger assumption that no formula A is both provable and disprovable.[10]

Still another idea that naturally leads to Nelson's system **N4** is the idea of atomicity of strong negation. If the provability conditions of a compound formula depend on the provability conditions of its components, then also the refutability conditions of a compound formula ought to depend on the refutability conditions of its components. But then, if negation ~ is to express falsity in the sense of refutability, one would need a negation normal form theorem to the effect that every negated formula ~A is provably equivalent to a formula nf (A) in which every occurrence of ~ stands immediately in front of an atomic formula. If, moreover, all literals (i.e. atoms and negated atoms) are to be treated on a par, because the refutability and the provability conditions of atomic formulas are independent of each other, then the result A^+ of replacing every occurrence of a negated atom ~p in nf (A) by a fresh atom p′ ought to leave the derivability relation unaffected. In Nelson's logics **N3** and **N4**, the wanted negation normal form theorem is obvious from the axiom schemata A1–A4. Moreover, by induction on proofs in **N4**, one can show that in fact atomicity of negation holds:

$$\Gamma \vdash_{\mathbf{N4}} A \quad \text{iff} \quad \Gamma^+ \vdash_{\mathbf{N4}} A^+$$

where $\Gamma^+ = \{B^+ | B \in \Gamma\}$.

This observation also leads to a simple derivation of the completeness of **N4** with respect to the class of all Nelson models from the completeness of positive logic with respect to the class of all intuitionistic Kripke models [see chapter 11]; cf. Gurevich (1977), Pearce (1991), or Rautenberg (1979).[11] Suppose that, in addition to the denumerable set Atom of prepositional variables, the set Atom′ = {p’ | p ∈ Atom} is also considered. An intuitionistic Kripke model for the language L = {⊃b, ∧, ∨) based on Atom ∪ Atom′ is a structure 〈**I**, ⊑, v〉, where 〈**I**, ⊑〉 is a partially ordered set and v is a valuation function mapping every propositional variable in Atom ∪ Atom′ to a subset of **I.** Moreover, persistence of atomic formulas is required: for every p ∈ Atom ∪ Atom′ and every t, u ∈ **I** such that t ⊑ u, t ∈ v(p) implies u ∈ v(p). The notion M, t ⊧ A (A is verified at t in M) is inductively defined as follows:

$$\mathcal{M}, t \models p \quad \text{iff} \quad t \in v(p),\ p \in Atom \cup Atom'$$

$$\mathcal{M}, t \models B \wedge C \quad \text{iff} \quad \mathcal{M}, t \models B \text{ and } \mathcal{M}, t \models C$$

$$\mathcal{M}, t \models B \vee C \quad \text{iff} \quad \mathcal{M}, t \models B \text{ or } \mathcal{M}, t \models C$$

$$\mathcal{M}, t \models B \supset_h C \quad \text{iff} \quad (\forall u \in \mathbf{I}) \text{ if } t \sqsubseteq u, \text{ then } \mathcal{M}, u \models B \text{ implies } \mathcal{M}, u \models C$$

Given such a model M = 〈**I**, ⊑, v〉, one can define a Nelson model M’ = 〈**I**, ⊑, v^+, v^-〉 by stipulating that for every p ∈ Atom ⊃ Atom’.

$$v^+(p) = v(p) \qquad \text{and} \qquad v^-(p) = v(p')$$

Lemma

For every ℒ-formula A.

(i) M′, $t \models^+ A$ iff M, t ⊧ A^+, and

(ii) M′, $t \models^- A$ iff M, t ⊧ $(\sim A)^+$.

Proof By simultaneous induction on the complexity of A. Consider only one case for the first claim, namely A= ~(B $\supset_h$ C):

$$\mathcal{M}', t \models^{+} \sim (B \supset_{b} C)$$

$$\text{iff} \quad \mathcal{M}', t \models^{+} B \text{ and } \mathcal{M}', t \models^{-} C$$

$$\text{iff} \quad [\mathcal{M}, t \models B^{+} \text{ (by the induction hypothesis for (i))} \text{ and } \mathcal{M}, t \models (\sim C)^{+} \text{ (by the induction hypothesis for (ii))}]$$

$$\text{iff} \quad \mathcal{M}, t \models (B^{+} \wedge (\sim C)^{+}) \; [= (\sim (B \supset_{b} C))^{+}]$$

Proposition

N4 is strongly complete with respect to the class of all Nelson models.

Proof Suppose Γ ⊬ A. Due to atomicity of negation in **N4**, $\Gamma^{+} \nvdash A^{+}$. Since positive logic is strongly complete with respect to the class of all intuitionistic Kripke models, there is such a model M= ⟨**I**, ⊑, v⟩ and t ∈ **I** such that for every B ∈ Γ, M, t ⊧ B^{+} and M, t A^{+}. The first part of the previous lemma guarantees that there is a Nelson model M' = ⟨**I**, ⊑, v^{+}, v^{-}⟩ such that for every B ∈ Γ, M', t $\models^{+}$ B and M, t $^{+}$ A. QED

A general definition of negation as falsity that is meant to encompass both intuitionistic negation and strong negation is suggested in Wansing (1999). Suppose that a single-conclusion consequence relation → over a formal language containing a unary connective * is given. In other words, for all formulas A, B and all finite sets of formulas Δ, Γ:

$$\textit{Reflexivity} \quad \vdash A \rightarrow A$$

$$\textit{Monotonicity} \quad \Gamma \rightarrow A \vdash \Gamma \cup \{B\} \rightarrow A$$

$$\textit{Cut} \quad \Gamma \cup \{A\} \rightarrow B, \Delta \rightarrow A \vdash \Gamma \cup \Delta \rightarrow B$$

A binary relation ← between finite sets of formulas and single formulas is called a single-conclusion *-refutation relation iff for all formulas A, B and finite sets Δ, Γ of formulas:

$$\star\textit{-reflexivity} \quad \vdash \star A \leftarrow A, \vdash A \leftarrow \star A$$

$$\star\textit{-cut} \quad \Delta \leftarrow A, \Gamma \cup \{\star A\} \leftarrow B \vdash \Delta \cup \Gamma \leftarrow B$$

Assume that → and ← are given as sequent calculi. If → is a single conclusion consequence relation, then * is a negation as falsity in → iff

(α)the relation ← defined by 'Δ ← A iff Δ → * A is a single-conclusion *-refutation relation

(β)for every formula A, not both ⊢ Ø → A and ⊢ Ø → *A

(γ)there is a formula A such that not both ø A → *A, ⊢ *A → A

If ← is a single conclusion *-refutation relation, then * is a negation as falsity in ← iff

(α′)the relation → defined by 'Δ → A iff Δ ← *A' is a single-conclusion consequence relation

(β′)for every formula A, not both ⊢ Ø ← A and ⊢ Ø ← *A

(γ′)there is a formula A such that not ⊢ A ← A.

If * satisfies both (α) and (α′) for a single-conclusion consequence relation → and a single-conclusion *-refutation relation ←, then negation as falsity is a vehicle for either keeping → and dispensing with ← or keeping ← and dispensing with →. Then both double negation introduction A → * *A and double negation elimination **A → A are derivable. Clearly, the relation ← defined by (α) is a single-conclusion *-refutation relation iff * satisfies ⊢ A → **A, and double negation introduction and analogues of (β) and (γ) are satisfied by strong negation in **N3** and **N4.**

Tennant (1999) also motivates negation in his system of intuitionistic relevant logic by considerations on both proofs and disproofs. However, whereas in Tennant's natural deduction system there are direct proofs, there are no direct disproofs of compound formulas. In this system there are no derivations not merely revealing the inconsistency of a premise set, but rather leading to the conclusion that a certain compound formula is refutable. A discussion of Tennant's approach can be found in Wansing (1999).

18.3.2. Negation as inconsistency

Gabbay (1988) defines a syntactic notion of negation as inconsistency. Suppose again that a single-conclusion consequence relation → over a formal language containing a unary connective * is given. The basic idea of Gabbay's definition is that the negation *A of a formula A is derivable from a set of premises F iff some undesirable formula B from a set of unwanted formulas θ* is derivable from Γ together with A. It is assumed that the logical object language either already contains or is conservatively extendible by a counterpart of the set-theoretical combination of premises. This counterpart is conjunction, ∧, governed by the following introduction rules:

$$(\rightarrow \wedge) \quad \Gamma \rightarrow A, \Gamma \rightarrow B \vdash \Gamma \rightarrow (A \wedge B)$$

$$(\wedge \rightarrow) \quad \Gamma \cup \{A, B\} \rightarrow C \vdash \Gamma \cup \{(A \wedge B)\} \rightarrow C$$

The unary operation * is then said to be a negation (as inconsistency) in → iff there is a non-empty set θ* of formulas which is not the same as the set of all formulas such that for every finite set Γ of formulas and every formula A:

$$\vdash \Gamma \rightarrow \star A \quad \text{iff} \quad (\exists B \in \theta^{\star}) \; (\vdash \Gamma \cup \{A\} \rightarrow B)$$

Moreover, θ* must not contain any theorems. If such a collection of unwanted formulas exists, it can always be chosen as {C| ⊢ Ø → *C}, since by (reflexivity), the latter set is non-empty, if * is a negation. The definition of negation as inconsistency can therefore be reformulated without appeal to θ*. Namely, * is a negation as inconsistency in → iff for every finite set Γ of formulas and every formula A:

$$\vdash \Gamma \rightarrow \star A \quad \text{iff} \quad \exists C (\vdash \varnothing \rightarrow \star C \;\&\; \vdash \Gamma \cup \{A\} \rightarrow C)$$

Negation in quite a few familiar logical systems can be shown to be a negation as inconsistency. In minimal, intuitionistic,

and classical logic, for example, θ* can be identified with the set of all explicit contradictions (A ∧ *A) in the respective language. In Gabbay and Wansing (1996) the notion of negation as inconsistency (alias inferential negation) is extended to a type of nonmonotonic inference relations between structured databases called structured consequence relations.
Every negation as inconsistency satisfies contraposition as a rule in the form:

$$\Gamma \cup \{A\} \rightarrow B \text{ implies } \Gamma \cup \{\star B\} \rightarrow \star A$$

Suppose Γ U {A} → B. Since *B → *B, by definition there is a C ∈ θ* such that {*B, B} → C. Applying (cut) one obtains Γ ∪ {*B, A} → C and hence Γ ∪ {*B} → *A. Therefore, strong negation in Nelson's constructive logics fails to be a negation as inconsistency. Also, every negation as inconsistency validates the Law of Excluded Contradiction, *(*A ∧ A). Since *A → *A, we have {*A, A} → B, for some B ∈ θ*. Hence, *A ¬ A → B, for some B ∈ θ*, and therefore Ø → *(*A ∧ A). Also double negation introduction ⊢ A → **A is provable:

$$\dfrac{\varnothing \rightarrow \star(A \wedge \star A) \quad \dfrac{A,\ \star A \rightarrow (A \wedge \star A)}{A,\ \star(A \wedge \star A) \rightarrow \star\star A}\ \text{contraposition}}{A \rightarrow \star\star A}\ \text{(cut)}$$

Moreover, we have ⊢ **A → *A:

$$\dfrac{\dfrac{\varnothing \rightarrow \star B \quad \dfrac{\dfrac{\dfrac{\star A \rightarrow \star A}{\{\star A,\ A\} \rightarrow B}\ \text{reformulated definition of negation}}{\{\star B,\ A\} \rightarrow \star\star A}\ \text{contraposition}}{}}{A \rightarrow \star\star A}\ \text{(cut)}}{\star\star\star A \rightarrow \star A}\ \text{contraposition}$$

(reflexivity)

for some B such that ⊢ Ø → *B. In the inverse direction we have:

$$\dfrac{\varnothing \rightarrow \star B \quad \dfrac{\dfrac{\star\star A \rightarrow \star\star A}{\{\star A,\ \star\star A\} \rightarrow B}}{\{\star B,\ \star A\} \rightarrow \star\star\star A}}{\star A \rightarrow \star\star\star A}$$

for some B such that ⊢ Ø → *B.

Observation

Suppose → is consistent in the sense that for no formula A of the underlying language, both Ø → A and Ø → *A are provable. Then * is a negation as inconsistency iff * satisfies contraposition as a rule, the Law of Excluded Contradiction, and double negation introduction.

Proof It has already been shown that the direction from right to left holds. For the direction from left to right suppose that ⊢ Γ → *A. Then ⊢ ∧ Γ → *A and by contraposition, ⊢**A → *∧ Γ. By the structural rules and the rules for ∧, ⊢ Γ, **A → *∧ Γ ∧∧ Γ, and since ⊢ A → **A, one may apply (cut) to obtain ⊢ Γ, A → *∧ Γ ∧ ∧Γ. Thus one may choose as θ* the set of all explicit contradictions A ∧ *A. This set θ* does not contain any theorems, because otherwise ⊢ Ø → C and ⊢ Ø → *C, for all formulas C. Suppose now that ⊢ Γ, A → (C ∧ *C). Then, by contraposition, ⊢ Γ, *(C ∧ *C) → *A, and since ⊢ Ø → *(C ∧ *C), an application of (cut) gives ⊢ Γ → *A QED

Obviously, not every recognized negation is a negation as inconsistency. Strong negation in Kleene's three-valued logic [see chapter 14], for example, fails to be a negation as inconsistency, because due to the absence of any theorems, the Law of Excluded Contradiction fails. The notion of negation as falsity is also non-trivial. In normal modal logic, for instance, necessity □ fails to be a negation as falsity, because ⊢ (p ∨¬ p) and ⊢ □ (p ∨¬ p) in contradiction of (β). In Wansing (1999), it is shown that every negation as inconsistency is a negation as falsity.

Observation

Every negation as inconsistency is a negation as falsity.

Proof It must be shown that the defined relation ← satisfies (*-reflexivity), (*-cut), (β), and (γ).

(*-reflexivity): *A ← A is immediate from *A → *A and the definition of ←. Since Ø → *(*A ∧ A) and {A, *A} → *A ∧ A, there is a C such that Ø → *C and {A} ∪ {*A} → C, hence A → **A and thus, by the definition of ←, A ← *A.

(*-cut): The (*-cut)-rule follows from the definition of ← and (cut) for →. Assume Δ ← *A and Γ ∪ {A} ← B. This means Δ → A and Γ ∪ {A} → *B. An application of (cut) gives Δ ∪ Γ → *B, which means Δ ∪ Γ → B, as required.

(β): Suppose that both Ø → A and Ø → *A for some A. Then there is a B ∈ θ* such that A → B. However, by (cut), → B, that is, θ* contains a theorem, quod non.

(γ): Suppose that for every formula A, A → *A and *A → A Then there is a B ∈ θ* such that A → B, that is Ø → *A. Applying (cut) to the latter and *A → A, gives Ø → A. But then, since θ* is non-empty, it must contain a theorem; a contradiction. QED

18.3.3. Negation as orthogonality

The interpretation of negation by means of the notion of orthogonality (or incompatibility) turns out to be very illuminating, because it enables a classification of different concepts of negation in terms of correlations between negation laws and algebraic as well as relational properties together with conditions on valuations. This semantic classification has been investigated in a series of papers by Dunn (1993, 1996, 1999).

Consider a partially ordered set (**A**, ≤, 0, **1**) with bounds **0, 1.** From an algebraic point of view, the elements of such a set may be seen as propositions. Intuitively, x ≤ y may be understood as 'x provably implies y.' In addition to ≤ one may consider another binary relation I on **A** and think of xIy as 'x is incompatible with y.' It is then natural to define two negations ¬ and − as follows:

$$x \leq \neg y \quad \text{iff} \quad xIy$$

$$x \leq -y \quad \text{iff} \quad yIx$$

Obviously, both notions of negation coincide if the not implausible assumption is made that incompatibility is a symmetric relation. By imposing constraints on ≤, ¬, and −, various concepts of negation can be defined. Interestingly, the defining properties can be correlated with relational properties of an incompatibility relation in so-called perp models (together with conditions on valuations). A perp frame is a structure (**I**, ⊑, ⊥), where the incompatibility relation ⊥ satisfies

$$(t \perp u \text{ and } t \sqsubseteq t') \text{ implies } t' \perp u$$

$$(t \perp u \text{ and } u \sqsubseteq u') \text{ implies } t \perp u'$$

A perp model is a structure M = (F, v), where F is a perp frame, and v is a valuation function persistent with respect to the relation ⊑. If X is a subset of **I**, let

$$t \perp X \quad \text{iff} \quad \text{for every } u \in X,\ t \perp u$$

$$X \perp t \quad \text{iff} \quad \text{for every } u \in X,\ u \perp t$$

One may then define ⊥ X = {t| t ⊥ X} and X⊥ = {t| X ⊥ t}. Negated formulas are evaluated according to the following clauses:

$$\mathcal{M}, t \vDash \neg A \quad \text{iff} \quad t \in {}^{\perp}|A|$$

$$\mathcal{M}, t \vDash -A \quad \text{iff} \quad t \in |A|^{\perp}$$

where |A| = {t ∈ **I**| M, t ⊢ A}. With these definitions, negated formulas are also persistent with respect to ⊑.
Dunn (1993, 1996) has shown that posets with bounds satisfying the properties listed below can be represented by perp frames satisfying the associated conditions:[12]

Negation	Posets	Perp models
Subminimal	$x \leq y \Rightarrow \neg y \leq \neg x$	$\lvert\neg A\rvert = \lvert A\rvert^{\perp}$
	$x \leq y \Rightarrow -y \leq -x$	$\lvert -A\rvert = {}^{\perp}\lvert A\rvert$
Galois	$x \leq \neg y \Leftrightarrow y \leq -x$	$\lvert\neg A\rvert = \lvert A\rvert^{\perp}$, $\lvert -A\rvert = {}^{\perp}\lvert A\rvert$
Minimal	$x \leq \neg y \Leftrightarrow y \leq \neg x$	⊥ symmetric
Intuitionistic	Minimal + $(x \leq y \ \&\ x \leq \neg y) \Rightarrow x \leq z$	⊥ irreflexive, symmetric
DeMorgan	Minimal + $\neg\neg x \leq x$	$\lvert A\rvert^{\perp\perp} = \lvert A\rvert$, ⊥ symmetric
Ortho	Intuitionistic + $\neg\neg x \leq x$	$\lvert A\rvert^{\perp\perp} = \lvert A\rvert$, ⊥ irrefl., symm.

The conditions on posets express inferential properties of negation. The negation logics defined by these properties are sound and, in view of Dunn's representation theorems, also complete with respect to the classes of perp models satisfying the associated conditions. Instead of considering partially ordered sets with bounds one may, of course, also study other algebraic structures, also for a logical object language richer than {¬, −}; see, for example, Dunn (1996). Since every subminimal negation satisfies the contraposition rule, however, extensions of subminimal negation like the negations investigated by La Palme Reyes et al. are unsuitable for representing predicate term negation.

18.3.4. Perfect negation

According to Avron (1999), a unary connective of a logic Λ is a perfect negation if it enjoys a certain syntactic property and, in addition, Λ is strongly normal in a certain sense. To state the syntactic property, first, various definitions are needed. Suppose Λ is presented as a single-conclusion sequent calculus (defined for premise multi-sets not necessarily satisfying monotonicity). Then its associated internal consequence relation $\to^{i}$ is defined by

$$A_1, \ldots, A_n \to^{i} A \quad \text{iff} \quad \vdash_{\Lambda} A_1, \ldots, A_n \to A$$

Λ's associated external consequence relation $\to^{e}$ is defined by:

$$A_1, \ldots, A_n \to^{e} A \quad \text{iff} \quad \to A_1, \ldots, \to A_n \vdash_{\Lambda} \to A$$

A unary connective * is said to be an internal negation for a consequence relation → iff the relation → is closed under

$$A, \Gamma \to \Delta \vdash \Gamma \to \Delta, \star A \qquad \text{and} \qquad \Gamma \to \Delta, A \vdash \star A, \Gamma \to \Delta$$

The existence of an internal negation forces a consequence relation to be a multiple-conclusion relation. A single-conclusion consequence relation → over a language with a unary connective * is said to be strongly symmetric with respect to * iff there exists a multiple-conclusion consequence relation →' defined over the same language such that

$$\Gamma \to' A \quad \text{iff} \quad \Gamma \to A$$

and * is an internal negation for →'.
If Λ is presented as a single-conclusion sequent calculus over a logical language containing a unary operation *, then, according to Avron, * is a perfect negation from the syntactic point of view if the internal consequence relation associated with Λ is strongly symmetric with respect to *. Intuitionistic negation and Nelson's strong negation fail to be perfect in this sense because every internal negation satisfies double-negation elimination and the contraposition rule. Indeed, Avron (1999, thm **4**) shows that if → is any consequence relation, then it is strongly symmetric with respect to * iff

(i) A →**A

(ii) **A → A, and

(iii) Γ, A → B implies Γ, *B → *A.

If the requirement of strong symmetry is relaxed in a certain way, strong negation still emerges as perfect from the syntactic point of view in a sense. Suppose → is a consequence relation defined over a language containing the unary operation *.
Define the multiple-conclusion consequence relation $\to^{S}$ by requiring that $A_1, \ldots, A_n \to^{S} B_1, \ldots, B_k$ iff for all **1** ≤ i ≤ n and **1** ≤ j ≤ k:

$$A_1, \ldots, A_{i-1}, \star B_1, \ldots, \star B_k, A_{i+1}, \ldots, A_n \to \star A_i$$

and

$$A_1, \ldots, A_n, \star B_1, \ldots, \star B_{j-1}, \star B_{j+1}, \ldots, \star B_k \to B_j$$

Then (Avron, 1999, propn 9)

(i) $\Gamma \to^{S} A$ implies Γ → A

(ii) Ø $\to^S$ A iff Ø → A, and

(iii) $\to^S$ is a conservative extension of → iff Γ, A → B implies Γ, * B → *A.

If one tries to see * as a negation in →, then, in Avron's view, $\to^S$ is induced by → in a natural way. The operation * in → is defined to be weakly symmetric iff it is an internal negation of $\to^S$. As Avron observes, * is weakly symmetric in → if A → **A and **A → A, and these conditions are satisfied by strong negation in Nelson's **N3** and **N4.**
A unary operation * in a logical system Λ is a perfect negation from the semantical point of view if Λ is strongly normal. For Avron (1999, p. 15), "a semantics is, essentially, just a set S of theories," since "the essence of a 'model' is given by the set of sentences which are true in it." A unary connective * is a (strong) semantic negation if in terms of validity in a model it reflects that every formula is either true or not true in a model and not both. Recall that a theory T is said to be consistent if there is no formula A such that both A and its negation are derivable from T. A theory T is complete if for every formula A, either A or its negation can be derived from T. If a theory is both consistent and complete, it is said to be normal.
Assuming that Λ is given by a single conclusion consequence relation, that the underlying language contains a unary operation * (considered to be a negation), and that for no formula A, both A and *A are provable, Avron presents various characterizations of Λ (1999, propn 26):

- •Λ is strongly complete iff whenever T ⊬ A there is a complete extension T′ of T such that T' ⊬ A.
- •Λ is weakly complete iff whenever Ø ⊬ A there is a complete theory T' such that T' ⊬ A.
- •Λ is strongly normal iff whenever T ⊬ A there is a complete and consistent extension T' of T such that T' ⊬ A.
- •Λ is weakly normal iff whenever Ø ⊬ A there is a complete and consistent theory T' such that T' ⊬ A.
- •Λ is c-normal iff every consistent theory in Λ has a complete and consistent extension.
- •Λ is strongly c-normal iff whenever T is consistent and T ⊬ A there is a complete and consistent extension T' of T such that T' ⊬ A.

Avron (1999, propn 27) observes that if Λ is finitary, then it is strongly complete iff for every theory T and all formulas A, B:

$$T, A \vdash_\Lambda B \quad \text{and} \quad T, \star A \vdash_\Lambda B \quad \text{implies} \quad T \vdash_\Lambda B$$

and that this condition is equivalent with the provability of A ∨ *A, if the underlying language contains a disjunction operation ∨ such that T, A ∨ B $\vdash_\Lambda$ C iff both T, A $\vdash_\Lambda$ C and T, B $\vdash_\Lambda$ C. Therefore, intuitionistic negation and any representation of predicate term negation as a unary connective are bound to be imperfect in Avron's sense. Indeed, the only negation among the unary connectives considered by Avron that emerges as perfect from both the syntactic and the semantic point of view is the Boolean negation of classical logic. Whereas intuitionistic logic and Kleene's three-valued logic are still strongly consistent and c-normal, and **N3** is still strongly consistent, **N4** enjoys none of the listed properties.

18.4. Epilogue

Although there is no general agreement on what is negation, whether it is a unary connective or an innersentential operation, whether contraposition as a rule holds for it or not, etc., it is accepted knowledge that there is more than one kind of negation, be it in the same syntactic type or in different categories. This insight to a large extent rests on the Aristotelian distinction between predicate denial and predicate term negation. Moreover, whereas the contradictory forming predicate denial can be represented by a contradictory forming sentential negation, the contrary forming predicate term negation can be represented by the strong negation in many-valued logics such as Nelson's constructive systems **N3** or **N4.**
The classification of kinds of sentential negation may be approached from various points of departure. One idea is to define a general non-trivial notion of negation in a system, meant to cover all recognized negations. This is-quite explicitly-the intention behind Gabbay's (1988) definition of negation in a system. The notion of negation as falsity generalizes Gabbay's suggestion; other definitions or additional requirements may lead to more restricted notions of negation. A classification of negation may, however, also be seen primarily as a means for identifying non-negations. Avron (1999, p. 21), for example, puts his criteria to such a use when he concludes that "[t]he negation of intuitionistic logic is not really a negation." A third aspect of a classificatory scheme is that it may help to recognize the interrelations between the items classified. In this respect, the semantic classifications of Dunn (1993, 1996, 1999) turn out to be particularly useful.

Suggested further reading

The classical reference as far as linguistic and philosophical aspects of negation are concerned is Horn's encyclopedic monograph (1989). This book also contains a careful introduction to
Aristotelian term logic. Another important reference to negation in Aristotelian term logic is Englebretsen (1981). More recent volumes devoted entirely to the study of negation are Wansing (1996) and Gabbay and Wansing (1999). The latter collection of research papers aims at providing a comprehensive account of negation from a logical, computational and philosophical point of view. A discussion of the notion of negation as finite failure to derive can be found in Fine (1989). The relation between monotonicity properties of natural language expressions and negative polarity items like 'anything' is dealt with, for example, in Zwarts (1996).
1 I shall not even try to survey this literature. Nor shall I deal with philosophically interesting negation related themes such as negative existentials, presupposition, or paraconsistency. Nothing at all will be said about the pragmatics or about psychological aspects of negation. Instead, the emphasis of this chapter will be on characterizing and classifying various

notions of negation.
2 Horn (1989, p. 38) remarks that “[t]he unique polar contrary and the unique immediate contrary of a given term will not in general coincide”.
3 Von Wright (1959, p. 5) criticizes Aristotle for his willingness to call both ‘John is well’ and ‘John is ill’ false, if the name ‘John’ does not denote anything: On this point Aristotle might be accused of obscuring a distinction which in other contexts he marks. I mean the distinction between the case when “x is P” is not true because x does not exist or because P cannot be “naturally predicated” of it, and the case when “x is P” is not true because not-P can be (truly) predicated of it. A convenient way of marking this distinction … would be to say that “x is P” is false only in the case when “x is not-P” is true.
4 There are exceptions such as ‘undecidable’ meaning ‘not decidable’.
5 In the sequel I shall not always pay attention to the mention/use distinction and omit quotation marks if misunderstandings are unlikely to occur.
6 From $x \leq y$ one obtains $x \wedge \neg y \leq y \wedge \neg y$, and since $y \wedge \neg y = 0$, and 0 is the least element, (18.1) implies $\neg y \wedge \neg x$.
7 Contraposible strong negation is studied in Nelson (1959).
8 A comprehensive study of four-valued logic, including Nelson's **N4**, can be found in Dunn (2000); see also Belnap (1977).
9 Other application areas in which the need for logics with more than one kind of negation arises include database theory, logic programming, and nonmonotonic reasoning; see, for instance, Wagner (1994), and Wansing (1995).
10 A more comprehensive discussion of the proof and disproof interpretation can be found in Wansing (1993).
11 An algebraic analysis of Nelson's logics can be found in Rasiowa (1974); axiomatic extensions of **N3** are investigated, for example, in Kracht (1998).
12 For DeMorgan and ortho negation we assume symmetry of $\perp$. Hence $\neg = -$ and we have M, t $\vdash \neg$ A iff for every u $\in$ **I**, t u implies M, u A. Since an inference A $\vdash$ B is defined to be valid in a perp model M = (**I**, $\sqsubseteq$, $\perp$, v) iff for every t $\in$ **I**, M, t $\models$ A implies M, t $\models$ B, the validity of double negation elimination in M amounts to the validity of $\Box\Diamond A \supset A$ in the modal model (**I**, , v). The latter formula is modally equivalent with the Sahlqvist formula $A \supset \Diamond\Box A$, and hence is first-order definable. It corresponds to

$$\text{(DNE)} \quad \forall t \exists u (t \not\perp u \wedge \forall s (u \not\perp s \supset t = s))$$

From the modal point of view, double negation introduction expresses the axiom schema $A \supset \Box\Diamond A$, which is known to correspond with the symmetry of the accessibility relation. Note that in the case of (DNE) we have shown correspondence but not completeness.

References

Almukdad, A. and Nelson, D. 1984: Constructible Falsity and Inexact Predicates , Journal of Symbolic Logic , (49) , 231 3.
Avron, A., Jr. 1999: Negation: Two Points of View , in Gabbay and Wansing (1999, pp. 3 22).
Belnap, N. D. 1977: A Useful Four-Valued Logic , in Modern Uses of Multiple-Valued Logic , J. M. Dunn and G. Epstein, eds., (D. Reidel , Dordrecht), 8 37.
van Benthem, J. F. A. K. 1991: Language in Action , (North-Holland , Amsterdam).
Dunn, J. M. 1993: Star and Perp: Two Treatments of Negation , in Philosophical Perspectives (Philosophy of Language and Logic) (7) , J. Tomberlin, ed., (Ridgeview , Atascadero, CA), 331 57.
Dunn, J. M. 1996: Generalized Ortho Negation , in Wansing (1996, pp.3 26).
Dunn, J. M. 1999: A Comparison of Various Model-Theoretic Treatments of Negation: A History of Formal Negation , in Gabbay and Wansing (1999, pp. 23 51).
Dunn, J. M. 2000: Partiality and its Dual , Studia Logica , (66) , 5 40.
Englebretsen, G. 1981: Logical Negation , (Van Gorcum , Assen).
Fine, K. 1989: The Justification of Negation as Failure , in Logic, Methodology and Philosophy of Science VIII , J. E. Fenstad I. T. Frolov and R Hilpinen, eds., (Elsevier , Amsterdam), 263 301.
Gabbay, D. M. 1988: What is Negation in a System , in Logic Colloquium ’86 , F. Drake and J. Truss, eds., (Elsevier , Amsterdam), 95 112.
Gabbay, D. M. and Wansing, H. 1996: What is Negation in a System? Negation in Structured Consequence Relations , in Logic, Action and Information , A. Fuhrmann and H. Rott eds., (de Gruyter , Berlin), 328 50.
Gabbay, D. M. and Wansing, H. (eds.) 1999: What is Negation , (Kluwer , Dordrecht).
Gurevich, Y. 1977: Intuitionistic Logic with Strong Negation , Studia Logica , (36) , 49 59.
Horn, L. 1989: A Natural History of Negation , (Chicago University Press , Chicago).
Kracht, M. 1998: On Extensions of Intermediate Logics by Strong Negation , Journal of Philosophical Logic , (27) , 49 73.
von Kutschera, F. 1969: Ein verallgemeinerter Widerlegungsbegriff fur Gentzenkalküle , Archiv für Mathematische Logik und Grundlagenforschung , (12) , 104 18.
La Palme Reyes, M., Macnamara, J., Reyes, G. E. and Zolfaghari, H. 1994: The Non-Boolean Logic of Natural Language Negation , Philosophia Mathematica , (2) , 45 68.
La Palme Reyes, M., Macnamara, J., Reyes, G. E. and Zolfaghari, H. 1999: Models for Non-Boolean Negations in Natural Languages Based on Aspect Analysis , in Gabbay and Wansing (1999, pp. 241 60).
López-Escobar, E. G. K. 1972: Refutability and Elementary Number Theory , Indagationes Mathematicae , (34) , 362 74.
Nelson, D. 1949: Constructible Falsity , Journal of Symbolic Logic , (14) , 16 26.
Nelson, D. 1959: Negation and Separation of Concepts in Constructive Systems , in Constructivity in Mathematics , A. Heyting, ed., (North-Holland , Amsterdam),208 25.
Pearce, D. 1991: n Reasons for Choosing N , Technical report 14/91, Gruppe fur Logik, Wissenstheorie und Information, Free University of Berlin.
Rasiowa, H. 1974: An Algebraic Approach to Non-classical Logics , (North-Holland , Amsterdam).
Rautenberg, W. 1979: Klassische und Nichtklassische Aussagenlogik , (Vieweg , Braunschweig).
Routley, R, 1974: Semantical Analyses of Propositional Systems of Fitch and Nelson , Studia Logica , (33) , 283 98.
Sommers, F. 1982: The Logic of Natural Language , (Clarendon Press , Oxford).
Tennant, N. 1999: Negation, Absurdity and Contrariety , in Gabbay and Wansing (1999, pp.199 222).
Thomason, R, 1969: A Semantical Study of Constructive Falsity , Zeitschrift für Mathematische Logik und Grundlagen der Mathematik , (15) , 247 57.

Wagner, G. 1994: Vivid Logic. Knowledge-Based Reasoning with Two Kinds of Negation , Lecture Notes in AI 764 , (Springer-Verlag , Berlin).
Wansing, H. 1993: The Logic of Information Structures , Lecture Notes in AI 681 , (Springer-Verlag , Berlin).
Wansing, H. 1995: Semantics-Based Nonmonotonic Inference , Notre Dame Journal of Formal Logic , (36) , 44 54.
Wansing, H. (ed.) 1996: Negation. A Notion in Focus , (de Gruyter , Berlin).
Wansing, H. 1999: Negation as Falsity: A Reply to Tennant , in Gabbay and Wansing (1999, pp. 223 38).
von Wright, G. H. 1959: On the Logic of Negation , Commentationes Physico-Mathematicae , (22) (Societas Scientarum Fennica , Helsinki), 2 30.
Zwarts, F. 1996: Facets of Negation , in Quantifiers, Logic and Language , J. van der Does and J. van Eijck, eds., (CSLI Publications , Stanford),385 421.

Cite this article

Wansing, Heinrich. "Negation." The Blackwell Guide to Philosophical Logic. Goble, Lou (ed). Blackwell Publishing, 2001. Blackwell Reference Online. 28 February 2008 <http://www.blackwellreference.com/subscriber/tocnode?id=g9780631206934_chunk_g978063120693421>

Bibliographic Details

The Blackwell Guide to Philosophical Logic
Edited by: Lou Goble
eISBN: 9780631206934
Print publication date: 2001
Cite this article

Table of Contents

The Blackwell Guide to Philosophical Logic