Information Percolation, Momentum, and Reversal*

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Abstract

The tendency of asset prices to trend over short horizons and revert over long horizons—momentum and reversal—can find an explanation within a simple, rational model. The key ingredient is word-of-mouth communication, which we introduce in the standard noisy rational expectations framework. Word-of-mouth communication accelerates the information revelation and generates momentum in asset returns. Due to social interactions, investors with heterogeneous trading strategies coexist in the marketplace—some agents are contrarians, others are momentum traders, yet momentum is not completely eliminated. Finally, word-of-mouth communication can propagate a rumor and a price run-up, followed by reversals once the rumor subsides.

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1 Introduction

One of the most pervasive facts in finance is the “momentum” effect. Prices keep moving in the same direction over an horizon of 6 months to 1 year. Not only is this effect economically significant—around 1% per month—but it also persists across countries and across asset classes, both in the cross section (Jegadeesh and Titman, 1993) and the time series (Moskowitz, Ooi, and Pedersen, 2012) of returns. Momentum is typically followed by a “reversal” effect (De Bondt and Thaler, 1985), whereby prices correct the past trend and move in the opposite direction over long-term horizons.

Short-term momentum and long-term reversal present a significant challenge to rational explanations—smart money investors should eliminate these predictable patterns. Behavioral theories therefore prevail: momentum traders, conservative investors, and attribution bias all generate momentum, while newswatchers, representativeness heuristic, and overconfidence all generate reversal.1 But the wide prevalence of momentum and reversal suggests that one specific behavioral trait is unlikely to explain these phenomena across different markets. Furthermore, the link between momentum and various measures of investor sentiment is weak, suggesting that sentiment-based theoretical models do not identify the main source driving momentum and reversal in the data.

In this paper we show that momentum and reversal can find a simple, rational explanation. Our model departs minimally from a traditional “noisy rational-expectations” model (Grossman and Stiglitz, 1980). We assume that information diffuses among a population of rational investors through interpersonal communication. Investors trade in centralized markets, but have to search for each other’s private information. That is, trading is centralized but information exchange is decentralized. Our model is therefore based on a social trait—interpersonal communication—that collectively affects the population of agents, as opposed to an individual behavioral trait.

Our model relies on a rational-expectations economy in which a large population of risk-averse investors trade a risky asset.2 Trading takes place at discrete dates until the asset’s final payoff (the fundamental) is realized. All investors are initially endowed with a private signal about the fundamental. Investors then meet randomly and exchange their information, as in the information percolation model developed by Duffie, Malamud, and Manso (2009).

Two learning channels arise in this model: agents learn from the informational content of prices and through word-of-mouth communication. The former channel is standard in the literature, whereas the latter channel is novel. We show that both learning channels are

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necessary components for understanding how momentum and reversal emerge. This result is particularly important, for it suggests that the informational role of prices—often dismissed in leading theories—is one of the key drivers of momentum and reversal.

Depending on the intensity at which agents meet with each other, information percolation produces momentum in stock returns. More precisely, when the frequency of meetings is low, stock prices exhibit reversals, a common outcome in noisy rational-expectations models (e.g., Wang 1993). We show, however, that as information percolation intensifies, the information flow through prices is sufficiently strong to eventually generate momentum in stock returns.

One fact about momentum is that it is easily identified from prices. This raises an obvious question: if rational investors are able to consistently detect momentum and trade profitably on it, then why does momentum persist? In our model, investors identify momentum from prices, some trade on it (buy when prices go up and sell when prices go down) and make profits. Nevertheless, these momentum traders do not completely eliminate momentum—momentum and momentum traders can coexist in the marketplace. A key element to generate this result is the heterogeneity in information endowments that results from interpersonal communication. Consequently, while everyone observes momentum, not everyone is a momentum trader—some agents are momentum traders, others are contrarians, and momentum therefore persists. This result distinguishes our model from one in which the precision of private information is exogenously assumed to increase over time; in such a model all investors would be contrarians, contradicting empirical evidence (Moskowitz et al., 2012).

Not only does information percolation generate momentum, but it is also a natural propagator of rumors. If private information contains a rumor, this rumor will circulate from one person to another, creating a disconnect between the stock price and the fundamental. This phenomenon arises in our model if we assume that investors are aware of the existence of an unobservable rumor. Word-of-mouth communication propagates this rumor causing a price overshooting. As trading approaches its final round, the fundamental value is about to be revealed and the rumor subsides, producing a price reversal.

Finally, while we derive our implications in a model with a terminal date, we show that our results extend to a fully dynamic setup. In particular, momentum obtains whether the asset pays a single liquidating dividend or an infinite stream of dividends.

Among the leading explanations for momentum and reversal, Daniel et al. (1998) and Hong and Stein (1999) are the ones most closely related to our paper, but our analysis differs in several respects. First, Daniel et al. (1998) rely on a parsimonious and simple model, the

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3In Daniel et al. (1998) agents are overconfident (they overestimate the precision of private information, which generate overreaction and reversals) and have an attribution bias (their confidence increases if public information corroborates what they have already assumed, which generates continuing overreaction and momentum). Instead, Hong and Stein (1999) argue that the interaction between momentum
core structure of which we borrow—yet we assume away any behavioral bias. Second, as in Hong and Stein (1999), we allow information to diffuse through the population of agents—yet, in our model this phenomenon naturally arises through word-of-mouth communication and produces two fundamentally distinct implications. First, we do not need to prevent investors from extracting information from prices. On the contrary, we need to allow agents to learn from prices to generate momentum—what eliminates momentum in Hong and Stein (1999) is actually a force driving momentum in our model. Second, we do not need to postulate different types of agents ab initio—these types arise endogenously. In particular, we assume that agents start off being identical individuals and then let word-of-mouth communication operate to generate a rich heterogeneity of types (some agents collect more information than others). In turn, this heterogeneity leads to different, endogenous investment strategies: depending on their information endowments, some agents act as (rational) contrarians, while other agents act as (rational) momentum traders.

The epidemiological approach to explaining momentum and reversal developed by Hong, Hong, and Ungureanu (2010) is similar to our approach. There are, however, significant differences. First, in Hong et al. (2010) agents agree to disagree and prices therefore have no informational content. Instead, in our model agents attempt to infer the information of other agents from prices. We believe that a credible theory for momentum should necessarily feature learning from prices—after all, what makes momentum a puzzle from a rational viewpoint is precisely that it can be identified in past prices and yet that it persists. Moreover, in Hong et al. (2010) information spreads like a disease—a sender transmits information to a receiver. In our paper we avoid this sender-receiver dichotomy by assuming that everyone receives information and exchanges this information at future meeting dates. On the technical side, the model of Hong et al. (2010) is difficult to solve even for a small number of agents, while most of our results are available in closed form for an infinite number of agents.4

Our work belongs to the information percolation literature (Duffie and Manso, 2007).5 While this literature focuses on settings in which markets are decentralized, our contribution

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4Beyond the particular scope of financial markets, several papers highlight the relevance of social networks in economics. Among others, Acemoglu, Bimpikis, and Ozdaglar (2010) investigate the effectiveness of dynamics of communication as an aggregator of dispersed information and Burnside, Eichenbaum, and Rebelo (2010) show how social dynamics explain the moves on the housing market. Other papers that emphasize the importance of social epidemics in economics are finance are Shive (2010) and Carroll (2006).

5The term “percolation” has been introduced to finance by Duffie and Manso (2007). They borrow this term from physics and chemistry, fields in which percolation concerns the movement and filtering of fluids through porous materials. In economics, it concerns the dissemination of information through large markets. See also Duffie et al. (2009), Duffie, Malamud, and Manso (2010a), and Duffie, Malamud, and Manso (2010b).
is to show that information percolation also has an important role to play in centralized markets. In particular, in this literature agents meet to trade, while in our model agents meet to share information.

Finally, our work originates from the “noisy rational-expectations” literature. This literature studies how prices aggregate information that is dispersely held by agents in the economy and recognizes that prices are a mechanism of information transmission. Paradoxically, to ensure the existence of an equilibrium, noise traders typically slow down information aggregation through prices (Grossman and Stiglitz, 1980). The informational role of prices is, therefore, necessarily imperfect, leaving room for other mechanisms of information transmission, such as word-of-mouth communication. Adding such a mechanism is our contribution to this literature.\footnote{A comprehensive investigation of word-of-mouth communication among traders in financial markets is provided by Shiller (2000), who writes: “Word-of-mouth transmission of ideas appears to be an important contributor to day-to-day or hour-to-hour stock market fluctuations”. In particular, Shiller (2000) argues that interpersonal communication, being an innate feature of humans, is more influential than traditional media or Internet. Stein (2008) describes conversations as being “a central part of economic life,” and Shiller and Pound (1989) provide evidence that direct interpersonal communication is an important determinant of investors decisions. Hong, Lim, and Stein (2000) show that stocks with greater analyst coverage exhibit more momentum, providing evidence in favor of the underreaction hypothesis. Hong et al. (2010) investigate, both theoretically and empirically, the relation between the serial correlation of returns and information diffusion based on word-of-mouth communication. Their conclusion is that a model of word-of-mouth communication is useful for understanding price momentum and dynamics around media events. Antweiler and Frank (2004) show that high message posting is associated with greater next day volatility, suggesting that a new and modern competitor to word-of-mouth is the transmission of information through Internet. For other contributions showing evidence that word-of-mouth communications play a significant role in financial markets, see Hong, Kubik, and Stein (2004), Feng and Seasholes (2004), Brown, Ivkovic, Smith, and Weisbenner (2008), Grinblatt and Keloharju (2001), Ivkovic and Weisbenner (2005) or Cohen, Frazzini, and Malloy (2008).}

2 Information Percolation

The concept of rational expectations (Grossman, 1981) captures the idea that stock prices aggregate information that is initially dispersed across investors. Due to noise trading, however, prices aggregate information only imperfectly, thus leaving room for other communication mechanisms. Word-of-mouth communication is an innate channel of information processing, which naturally complements information processing from prices. Other ways of communication may be considered: public announcements, for instance, also contribute to spread information. In a rational expectation setup, however, public announcement do not generate trading heterogeneity. At the core of the momentum puzzle is the fact that some investors follow the trend, others are contrarians, and yet momentum persists (Moskowitz et al., 2012). We believe that word-of-mouth communication provides a plausible explanation for momentum as it preserves the concept of trading heterogeneity.
To model word-of-mouth communication among investors, we use the information percolation mechanism of Duffie et al. (2009). Specifically, we offer a model of centralized trading—a noisy rational expectations equilibrium—and decentralized information gathering—information percolation. In equilibrium, information percolation has two central effects. First, it impacts the speed at which information flows through prices, ultimately affecting the dynamics of equilibrium prices. Second, as information percolates through the population, agents become heterogeneous with respect to their information endowment—some become more informed than others. As a result, investors implement heterogeneous trading strategies. In this section, we elaborate on the resulting information heterogeneity and show how information percolation generates tractable cross-sectional dynamics.

2.1 Information Setup

We build an economy with four trading dates, indexed by \( t = 0, 1, 2, 3 \), and a final liquidation date, \( t = 4 \). The economy is populated by a continuum of investors indexed by \( i \in [0, 1] \). There is a risky security with payoff \( \tilde{U} \) realized at the liquidation date. The payoff of this security is unobservable and follows a normal distribution with zero mean and precision \( H \).

Immediately prior to each trading session, each investor \( i \) obtains a private signal about the asset payoff, \( \tilde{z}_i^t \):

\[
\tilde{z}_i^t = \tilde{U} + \tilde{c}_i^t
\]

where \( \tilde{c}_i^t \) is distributed normally and independently of \( \tilde{U} \), has zero mean, precision \( S \), and is independent of \( \tilde{c}_j^k \) if \( k \neq i \) or \( j \neq t \).

The precision of private signals is constant over time and is the same across investors, which allows us to perfectly isolate the effects of information diffusion. We introduce a mechanism whereby agents accumulate signals: from date \( t = 0 \) onward, we assume that private information diffuses across the population of agents through word-of-mouth communication. Individual precisions therefore increase over time and become heterogeneous across investors.

To formalize this idea, we use the setup of Duffie et al. (2009). Between time 0 and time 3, we assume that agents meet each other randomly. Meetings take place at Poisson arrival times with intensity \( \lambda \). This parameter—the only parameter we add to an otherwise standard equilibrium model—summarizes the behavior of the whole social network. When agents meet, they exchange truthfully their initial signal and other signals that they received during previous meetings (if any). Because signals are normally distributed, an agent’s total number

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7The number of four trading dates is the minimum necessary structure to show our effects; the model can be easily extended to an arbitrary number of \( N \) trading dates.

8We refer to the precision of a random variable as the inverse of its variance.
of signals and her posterior expectation of the fundamental completely summarize her private information; these two numbers are what agents actually exchange when they meet.

To illustrate how information percolation works, pick two agents—say $D$ and $J$—out of the crowd. $D$ and $J$ start with one signal. Suppose the first time they meet someone, they meet each other. They exchange their signals truthfully and therefore end up with 2 signals after the meeting. Suppose further that $D$ meets someone else, say $E$ who also has two signals (i.e., $E$ also met someone before). Since $D$ and $E$ are part of an infinite crowd of agents, the person that $E$ has met cannot be $J$, it must be someone else—meetings do not overlap.\footnote{In other words, there is a zero probability that the set of agents that $D$ has met before time $t$ overlaps with the set of agents that $J$ has met before time $t$. This eliminates the concern that we are introducing persuasion bias in the terms of Demarzo, Vayanos, and Zwiebel (2003): an agent might share her signals to another agents who passes those signals at subsequent meetings to other agents and maybe the same signals will come back to the first agent—without her knowledge. The infinite mass of agents prevents this double accounting of signals to happen, since the probability for an agent to meet in the future precisely those agents who got her signals is zero. Thus, for every pair $(i, j)$ of agents, their signal sets are always disjoint.}

Hence, after the meeting, $D$ and $E$ both part with four signals each. Signals keep on adding up randomly in the exact same way for every agent in the economy.

Meetings are idiosyncratic and therefore produce a rich heterogeneity of information across agents—while agents start off holding one signal, they end up holding heterogeneous numbers of signals as soon as they meet each other. This percolation-driven heterogeneity is summarized by the cross-sectional distribution of the number of signals. Formally, say that between time $t - 1$ and $t$ each agent $i$ collects a number $\omega_i^t$ of signals, including the one received at time $t$, and excluding the signals received up to time $t - 1$. For the rest of this paper, when we mention the number of signals that an agent has at time $t$, we refer to the number of additional signals, thus excluding the ones collected up to time $t - 1$.\footnote{Notice that both the distribution over the total number of signals and the distribution over additional signals may be equivalently used; we choose to use distribution of additional signals because it helps us better separate and understand the effects of information percolation on the equilibrium price and trading strategies.}

Since agents are initially endowed with a single signal, the initial distribution of signals has 100% probability mass at $n = 1$ and 0% probability mass at $n > 1$. As information diffuses (at dates $t > 0$), the distribution $\pi_t$ takes positive values over $\mathbb{N}^*$. For example, an agent who did not meet anyone between $t - 1$ and $t$ receives only one additional signal at $t$ and thus is of type $n = 1$. An agent who collected ten signals between $t - 1$ and $t$ receives one more signal at $t$ and thus is of type $n = 11$, and so on. Following Duffie et al. (2009), the cross-sectional distribution of the number of signals satisfies the following Boltzmann equation\footnote{\textcopyright Springer Science+Business Media, LLC 2013.}:

$$\frac{d}{dt} \pi_t(n) = \lambda \pi_t \ast \pi_t - \lambda \pi_t = \lambda \sum_{m=1}^{n-1} \pi_t(n-m)\mu_t(m) - \lambda \pi_t(n)$$

where $\ast$ denotes the discrete convolution product and $\mu$ represents the cross-sectional dis-
tribution of the total number of signals, which we define in Appendix A.1. The summation term on the right hand side in (1) represents the rate at which new agents of a given type are created. The second term in (1) captures the rate at which agents leave a given type (e.g., when an agent of type 10 meets someone, she is no longer of type 10).

This setup has the advantage of leading to a closed-form solution for the cross-sectional distribution $\pi_t$ of the number of signals. It is given in the following proposition, the proof of which is provided in Appendix A.1.\footnote{Although we present here closed-form solutions for the distribution of signals, we can also compute this distribution numerically through inverse Fourier transform in a very efficient fashion. We provide the details on the numerical procedure in Appendix A.1.2.}

**Proposition 1.** The probability density function over the additional number of signals collected by each agent between $t-1$ and $t$, with $t \geq 1$, is given by

$$
\pi_1(n) = e^{-n\lambda} \left( e^\lambda - 1 \right)^{n-1}
$$

$$
\pi_2(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \left[ \binom{i}{j-1} \right] \sum_{k=0}^{j} \binom{i+j+k-1}{i+j+k} e^{-i\lambda-(j+1)\lambda} \left( e^\lambda - 1 \right)^i e^{-j\lambda-(k+1)\lambda} \left( e^\lambda - 1 \right)^j
$$

$$
\pi_3(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} \binom{j}{k-1} \sum_{l=0}^{k} \binom{i+j+k+l-1}{i+j+k+l} e^{-(i+j)\lambda-(k+l)\lambda} \left( e^\lambda - 1 \right)^i
$$

Figure 1 illustrates the evolution of the cross-sectional distribution $\pi_t(n)$ of signals for a given value of the meeting intensity $\lambda$: the upper and lower panel depict the distribution at time 1 and time 2, respectively. The way this distribution changes through time has two important implications. First, the mass of the distribution gradually shifts towards larger number of signals. As a result, the average number of signals, and therefore the average precision, increases over time. Second, while the distribution is initially concentrated at 1—each agent starts off with one signal—it rapidly spreads to reflect the growing heterogeneity of information in the population. This heterogeneity varies itself through time, as apparent in the upper and lower panel of Figure 1.\footnote{Other aspects are worth noting. First, the probability mass at $n = 1$ stays constant for all periods (it equals $e^{-\lambda}$, roughly 37% in this case); these are investors who do not meet anyone during the last period and consequently end up with only the usual signal received at the trading date. Second, at time 2 there is zero probability mass for $n = 2$. The reason is that no investor could gather 2 additional signals between dates 1 and 2, since that would require meeting an investor with only one signal during that period, and no such investor exists.}

Finally, because signals are continuously reshuffled through the population, agents get from their peers signals that have already been used in previous trading sessions. The price therefore partially reflects this information—the signal is partially “stale”. By the law of large numbers, the price only aggregates information about the fundamental $\tilde{U}$ and idiosyncratic noise $\tilde{\epsilon}_t$ in individual signals wipes out in the aggregate. As a result, agents consider signals
Figure 1: Evolution of Cross-Sectional Densities

The Figure depicts the evolution of the probability density function of the number of additional signals through time when agents are endowed with one signal at each period. Each graph depicts $\pi$ at time $t = 1$ and $t = 2$, respectively, with $\lambda = 1$.

that they receive genuinely new, even though their information content decreases over time.

We now use the information percolation mechanism and build a model of trading in which information diffuses through word-of-mouth communication.

3 Equilibrium Prices and Optimal Trading Strategies

We embed the information percolation mechanism of the previous section within a standard model of trading à la Grossman and Stiglitz (1980). While our model preserves the insights of a noisy rational expectations equilibrium, it produces two additional, central implications. First, information percolation accelerates the flow of information through prices and thereby generates momentum. Second, as information percolates, agents receive heterogeneous endowments of information and therefore implement heterogeneous investments strategies: some agents are trend-followers, others are contrarians.

3.1 Setup

Investors have exponential utility with common coefficient of absolute risk aversion $1/\gamma$. The asset payoff is realized and consumption takes place at time $t = 4$, while trading takes place at times $t = 0, 1, 2, 3$. Each investor $i$ is endowed at time $t = 0$ with a quantity of the risky asset represented by $X^i$. At each trading date, investor $i$ chooses a position in the risky asset,
\( \tilde{D}_t \), to maximize expected utility of terminal wealth, denoted by \( \tilde{W}_4 \):

\[
\max_{\tilde{D}_t} \mathbb{E} \left[ e^{-\frac{1}{\gamma} \tilde{W}_4} \left| \tilde{F}_t \right| \right]
\]

subject to

\[
\tilde{W}_4 = X^i \tilde{P}_0 + \sum_{j=0}^{2} \tilde{D}_j \left( \tilde{P}_{j+1} - \tilde{P}_j \right) + \tilde{D}_3 \left( \tilde{U} - \tilde{P}_3 \right)
\]

where \( \tilde{F}_t \) represents the information set of investor \( i \) at time \( t \). This information set comprises: (i) private signals received at each date and collected through the information percolation mechanism described in Section 2, and (ii) prices (endogenously determined in equilibrium and denoted by \( \tilde{P}_t \)) as public signals.\(^{13}\)

The aggregate per capita supply of the risky asset at time \( t = 0 \), \( \tilde{X}_0 = \int_0^1 X^i di \), is normally and independently distributed with zero mean and precision \( \Phi \). New liquidity traders enter the market in trading sessions \( t = 1, 2, 3 \). The incremental net supply of liquidity traders, \( \tilde{X}_t \), is normally distributed with zero mean and precision \( \Phi \).

The noisy supply assumption is a technique commonly used in rational expectations models to prevent asset prices from fully revealing the final payoff \( \tilde{U} \). We adopt a random walk specification for the noisy supply (or, equivalently, we assume that increments in noisy supply are iid)\(^{14}\). Under this specification, any pattern in the correlation of returns depends on the time distribution of private information. We can, therefore, isolate the link between the diffusion of information and the serial correlation of returns.\(^{15}\) Moreover, this specification makes sure that our results are not artificially driven by the persistence of noise traders’ demand.\(^{16}\)

The solution method for finding a linear, partially revealing rational expectations equilibrium is standard and is relegated to Appendix A.2. We describe the equilibrium below.

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\(^{13}\)Our model bears similarities with Brennan and Cao (1997), with the main difference that we embed an information diffusion mechanism. We focus on a single asset economy, featuring several trading dates and a final liquidation date, in order to keep the setup comparable with leading momentum theories, such as Daniel et al. (1998) and Hong and Stein (1999).

\(^{14}\)i.i.d. incremental changes in noisy supply are likely to happen when time between consecutive trading dates is small.

\(^{15}\)Other specifications, such as an an AR(1) noise trading process, give qualitatively similar results.

\(^{16}\)Liquidity, or noise traders, can actually be rational, as Cespa and Vives (2012) point out. They can be considered risk-averse, competitive hedgers, who receive a random shock to their endowment. Upon receiving that shock, these infinitely risk-averse hedgers get rid of the endowment shock in the market place, producing a similar effect that characterizes a model with noise traders. This would also bring our model closer to Moskowitz et al. (2012), who find that speculators profit from momentum strategies at the expense of hedgers.
3.2 Equilibrium with Information Percolation

We first introduce notation and terminology for further use. At each date \( t \), agent \( i \) receives \( \omega_i^t \) new signals. From Gaussian theory, it is sufficient to keep track of her average incremental signal, a single signal with precision \( S_{\omega_i^t} \), which we denote by \( \tilde{Z}_i^t \):

\[
\tilde{Z}_i^t = \tilde{U} + \tilde{\epsilon}_i^t, \quad \text{where } \tilde{\epsilon}_i^t \sim N \left( 0, \frac{1}{S_{\omega_i^t}} \right)
\]

For convenience, we group all the information concerning the random variables in the present setup in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Mean</th>
<th>Precision</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental</td>
<td>( \bar{U} )</td>
<td>0</td>
<td>( H )</td>
<td></td>
</tr>
<tr>
<td>Per capita supply</td>
<td>( \bar{X}_0 ) = \int_0^1 X^i , di</td>
<td>0</td>
<td>( \Phi )</td>
<td>at date ( t = 0 )</td>
</tr>
<tr>
<td>Liquidity traders</td>
<td>( \bar{X}_t )</td>
<td>0</td>
<td>( \Phi )</td>
<td>at date ( t = 1, 2, 3 )</td>
</tr>
<tr>
<td>Private signals</td>
<td>( \tilde{Z}_0^t = \bar{U} + \tilde{\epsilon}_0^t ) (for ( \tilde{\epsilon}_0^t ))</td>
<td>0</td>
<td>( S )</td>
<td>at date ( t = 0 )</td>
</tr>
<tr>
<td>Private signals</td>
<td>( \tilde{Z}_t^t = \bar{U} + \tilde{\epsilon}_t^t ) (for ( \tilde{\epsilon}_t^t ))</td>
<td>0</td>
<td>( S_{\omega_i^t} )</td>
<td>at date ( t = 1, 2, 3 )</td>
</tr>
</tbody>
</table>

Table 1: Random variables in the present model

Another important statistic in this economy is the cross-sectional average of the number of additional signals at time \( t \), \( \Omega_t \equiv \sum_{\omega_t \in N} \pi_t (\omega_t) \omega_t \), where \( \pi_t (\omega_t) \) has been defined and computed in Section 2. At time 0, investors start with a single signal and therefore \( \omega_0^i = \Omega_0 = 1 \), \( \forall i \in [0, 1] \).

We finally define some variables from Theorem 1 below, needed for the description of the equilibrium solution. The conditional precision of agent \( i \) about the final payoff \( \bar{U} \), given all available information (private and public signals), is denoted by \( K_i^t \) and defined in (4). The cross-sectional average of conditional precisions over the entire population of agents is denoted by \( K_t \) and defined in (5). The conditional expectation of agent \( i \) about the final payoff \( \bar{U} \) is denoted by \( \tilde{\mu}_i^t \) and defined in (6). In (7), we define 2 types of informational advantages. First, \( a_i^t \) represents the marginal informational (dis)advantage arising from private signals gathered from time \( t - 1 \) to \( t \). Second, \( A_i^t \) represents the cumulative informational (dis)advantage arising from private signals gathered from time 0 to \( t \). These informational (dis)advantages are relative measures, comparing the individual precision of investor \( i \) with the average precision across the population of investors (marginal and cumulative). Finally, we normalize public signals (prices) under the form (8).

Theorem 1, the proof of which is given in Appendix A.2, describes the risky asset prices and investor asset demands at each date in a noisy rational expectations equilibrium with information percolation.
Theorem 1. There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price, $\tilde{P}_t$, and individual asset demands, $\tilde{D}_i^t$, for $t = 0, \ldots, 3$, are given by:

$$\tilde{P}_t = \frac{K_t - H}{K_t} \tilde{U} - \sum_{j=0}^{t} \frac{1 + \gamma^2 S \Omega_j \Phi}{\gamma K_t} \tilde{X}_j$$ \hspace{1cm} (2)

and

$$\tilde{D}_i^t = \gamma \left[ \sum_{j=0}^{t} (S \omega_i^j \tilde{Z}_j^i - S \Omega_j \tilde{Q}_j) - A_i^t \tilde{P}_t \right]$$ \hspace{1cm} (3)

where

$$K_i^t \equiv \text{Var}^{-1} \left[ \tilde{U} \mid \tilde{F}_t^i \right] = H + \sum_{j=0}^{t} S \omega_i^j + \sum_{j=0}^{t} \gamma^2 \Phi S^2 \Omega_j^2$$ \hspace{1cm} (4)

$$K_t \equiv \sum_{\omega \in \mathbb{N}} K_i^t(\omega) \pi_t(\omega) = H + \sum_{j=0}^{t} S \Omega_j + \sum_{j=0}^{t} \gamma^2 \Phi S^2 \Omega_j^2$$ \hspace{1cm} (5)

$$\bar{\mu}_i^t \equiv \mathbb{E} \left[ \tilde{U} \mid \tilde{F}_t^i \right] = \frac{1}{K_t} \sum_{j=0}^{t} \left[ S \omega_i^j \tilde{Z}_j^i + \gamma^2 S^2 \Phi \Omega_j \tilde{Q}_j \right]$$ \hspace{1cm} (6)

$$a_i^t \equiv S \omega_i^t - S \Omega_t, \quad A_i^t \equiv K_i^t - K_t$$ \hspace{1cm} (7)

$$\tilde{Q}_i \equiv \tilde{U} - \frac{1}{\gamma S \Omega_t} \tilde{X}_t$$ \hspace{1cm} (8)

The optimal trading strategy of the individual investor $i$ at time $t = 1, 2, 3$, denoted by $\Delta \tilde{D}_i^t \equiv \tilde{D}_i^t - \tilde{D}_{i-1}^t$ takes the following form:

$$\gamma \left[ a_i^t \left( \tilde{Z}_i^t - \tilde{P}_{t-1} \right) + S \Omega_t \left( \tilde{Z}_i^t - \tilde{P}_{t-1} \right) - \frac{K_t}{1 + \gamma^2 S \Phi \Omega_t} \left( \tilde{P}_t - \tilde{P}_{t-1} \right) - A_i^t \left( \tilde{P}_t - \tilde{P}_{t-1} \right) \right]$$ \hspace{1cm} (9)

3.2.1 Interpretation of Results Without Information Percolation

We first study the results of Theorem 1 in an economy without information percolation ($\lambda = 0$). In this case we have $\omega_i^t = \Omega_t = 1, a_i^t = A_i^t = 0$, and $K_i^t = K_t$, for any $i \in [0, 1]$ and for $t = 0, 1, 2, 3$.

Equation (2) shows that the asset price is a linear function of the final payoff and supply shocks, as is customary in the noisy rational expectations literature. Equation (3) shows that the optimal demand of investor $i$ in period $t$ depends on private signals ($\tilde{Z}_j^i$) and public
signals ($\tilde{Q}_j$). Without information percolation, the demand boils down to

$$\tilde{D}_i^t = \gamma S \sum_{j=0}^{t} \left( \tilde{Z}_j^i - \tilde{Q}_j \right)$$

Investor $i$ compares private signals with public signals, demands more of the risky asset if she observes relatively higher private signals ($\tilde{Z}_j^i > \tilde{Q}_j$), and trades more aggressively if risk aversion ($1/\gamma$) is low or the precision of private information ($S$) is high.

The optimal trading strategy of investor $i$ at time $t$, stated in Equation (9) has four terms. Without information percolation, the first and the last term are equal to zero (investors have homogeneous information endowments and thus $a_i^t = A_i^t = 0$). In this case, only two terms drive the optimal trading strategy: the second and the third term in (9). The second term, which becomes $\gamma S \left( \tilde{Z}_i^t - \tilde{P}_{t-1} \right)$ without information percolation, shows how investor $i$ trades based on private information: if the investor observes a positive signal, higher than the past price, she increases her position in the asset. Furthermore, a lower risk aversion or a higher private signal precision amplifies this effect. The third term, which becomes $-\frac{\gamma K_i}{1+\gamma S} \left( \tilde{P}_t - \tilde{P}_{t-1} \right)$ without information percolation, shows how investor $i$ trades based on public information: if the price today is higher than the past price, there is a chance that the supply is low today, hence the investor decreases her position to accommodate the supply.

To measure the trading behavior of investor $i$, we follow Brennan and Cao (1997) and compute the expected trade of investor $i$ conditional on the price change at time $t$, $\Delta \tilde{P}_t$. This measure can be computed by the econometrician, who does not observe supply shocks nor private signals:

$$E \left[ \Delta \tilde{D}_i^t | \Delta \tilde{P}_t \right] = \gamma S E \left[ \tilde{Z}_i^t - \tilde{Q}_t | \Delta \tilde{P}_t \right]$$

$$= \gamma S E \left[ \tilde{U} + \tilde{e}_i - \tilde{U} + \frac{1}{\gamma S} \tilde{X}_t | \Delta \tilde{P}_t \right]$$

$$= E \left[ \tilde{X}_t | \Delta \tilde{P}_t \right]$$

$$= m_t \Delta \tilde{P}_t$$

where the sign of the coefficient $m_t$ determines the investment style. When this coefficient is positive, an agent tends to buy after price increases and sell after price decreases. Thus, the econometrician observes momentum trading when $m_t$ is positive and contrarian trading when $m_t$ is negative. The following Proposition shows that, in an economy with homogeneous information endowments, all investors find it optimal to be contrarians.

**Proposition 2.** In an economy without information percolation, all investors adopt contrarian strategies, i.e., their trades are negatively correlated with the price change in the current
Proof. Following from (2) and (10), the conditional expected trade can be written as

\[
E \left[ \Delta \tilde{D}_t \mid \Delta \tilde{P}_t \right] = -\frac{1 + \gamma^2 S\Phi \var \left( \tilde{X}_t \right)}{\gamma K_t \var \left( \Delta \tilde{P}_t \right)} \Delta \tilde{P}_t
\]

It is straightforward to see that \( m_t < 0 \). Thus, all investors are contrarians. □

Proposition 4 makes an interesting point. Suppose homogeneous information endowments generate momentum in a noisy rational expectations model. Proposition 4 counterfactually shows that no investor would find it optimal to extract momentum profits by following the trend. Instead, Moskowitz et al. (2012) document that “speculators’ position load positively on time series momentum, while hedgers’ positions load negatively on it”. Any theory of momentum must be consistent with this observed heterogeneity in trading positions.

Although momentum may arise in a standard rational expectations model (as we show in Section 4.2.1), we must yet explain why some traders engage in momentum trading and others in contrarian trading. Below we show how information percolation helps generate this prediction.

### 3.2.2 Interpretation of Results With Information Percolation

This Section describes a mechanism through which investors with heterogeneous trading strategies optimally coexist in the market—some agents are momentum traders, others are contrarians. Furthermore, we show how trading on momentum crucially depends on the timing of private information.

Information percolation generates two simultaneous effects. First, when \( \lambda > 0 \), the average precision of private information, \( S\Omega_t \), increases over time as agents accumulate more signals through private meetings. Because prices directly depend on the average precision of private information, as shown in (2), a gradual increase in precision ultimately affects the price dynamics. In Section 4, we analyze the implication of this effect for the serial correlation of asset returns.

Second, information percolation generates heterogeneity in information endowments. In this case, at trading dates \( t = 1, 2, 3 \), the marginal and/or the cumulative informational advantages, \( a^i_t \) and \( A^i_t \), are different from 0. In turn, informational advantages impact optimal trading strategies; the first and last term in the optimal trading strategy (9) become relevant, in the following ways.
Consider an investor who has a *marginal* informational advantage at time $t$, i.e., $a_t^i > 0$ and who observes a private signal relatively higher than the past price. The first term in the optimal trading strategy, $\gamma a_t^i (\tilde{Z}_t^i - \tilde{P}_{t-1})$, shows that this investor increases her position in the risky asset, adding up to the position coming from the second term in (9). Because the investor has a marginal informational advantage, she feels more confident about her private signals received at time $t$, and thus trades more aggressively on private information.

Consider an investor who has a *cumulative* informational advantage at time $t$, i.e., $A_t^i > 0$ and who observes a price today higher than the past price. According to the last term in (9), $-\gamma A_t^i (\tilde{P}_t - \tilde{P}_{t-1})$, the investor decreases her position in the risky asset, adding up to the (already negative) position coming from the third term in (9). Because the investor has an overall information advantage, she is better able to infer the supply shock from the price signal and to accommodate it, and thus trades more aggressively on public information.

Investors with marginal and cumulative informational advantages trade more aggressively than the average investor, both on private and public information. It is yet unclear whether these adjustments in the trading strategy make the investor a trend follower or a contrarian—the first two terms in (9) represent momentum trading,\(^{17}\) whereas the last two terms in (9) represent contrarian trading. Depending on which effect dominates, the investor optimally chooses to be a contrarian or a trend-follower. The converse applies for an investor with marginal and cumulative information *disadvantages.*

Figure 2 helps us to be explicit about the tradeoff between being a trend-follower or a contrarian and to illustrate the heterogeneity in trading strategies generated by information percolation. The Figure depicts the “investment style coefficient” $m_1$ for different values of the meeting intensity $\lambda$.\(^{18}\) For $\lambda = 0$ all investors are contrarians, the result of Proposition 4. When $\lambda$ is positive, however, investors adopt heterogeneous trading strategies. Using the probability density function defined in Section 2, we consider three investors types: (i) the 5% percentile *less informed investor* (dotted red line), (ii) the *average informed investor* (black solid line), and (iii) the 95% percentile *better informed investor* (dashed blue line). The shaded area between the 5% percentile and 95% percentile represent 90% of the investor population. Figure 2 shows that, for higher values of $\lambda$, the relatively well-informed investors become momentum traders (dark gray area in the graph). The marginal informational advantage is sufficiently high for these investors to trade aggressively on their private information.\(^{19}\)

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\(^{17}\)It can be easily shown that $\tilde{Z}_t^i - \tilde{P}_{t-1}$ is positively correlated with $\Delta \tilde{P}_t$.

\(^{18}\)The analysis is simpler at time $t = 1$ because the marginal and cumulative informational (dis)advantages are equal.

\(^{19}\)At this point, an important difference arises between our explanation of momentum and the one provided by Hong and Stein (1999). Like Hong and Stein (1999), we recognize that a credible theory for momentum must necessarily rely on momentum traders, but in our case they arise endogenously through word-of-mouth communication. Thus, we bring forth an information diffusion mechanism which naturally generates
Figure 2: Information Percolation and Heterogeneous Trading Strategies at $t = 1$

The Figure depicts the coefficient of the price difference $\Delta \tilde{P}_1$ in the expectation $\mathbb{E} \left[ \Delta \tilde{D}_1 | \Delta \tilde{P}_1 \right]$ at time $t = 1$, i.e., $m_1$. A positive coefficient means momentum trading, whereas a negative coefficient means contrarian trading. There are three lines. The black solid line corresponds to the average informed investor, the red dotted line corresponds to the 5% percentile less informed investor, and the blue dashed line corresponds to the 95% percentile better informed investor. Thus, the gray area represents 90% of the population of investors. The sign of the “investment style coefficient” $m_1$ defines momentum or contrarian trading.

The analysis becomes more involved at time 2, because the marginal and informational advantages need not be equal (some traders can have a marginal advantage but not a cumulative one, or vice-versa). Figure 3 shows that this extra layer of analysis adds depth to our understanding of momentum trading. Consider first an investor who was relatively less informed at time 1 (panel a). When $\lambda$ is high, there is a high chance for a large marginal advantage at time 2. Consequently most investors who had a low information advantage at time 1 (and thus were contrarian traders) adopt trend following strategies at time 2. On the contrary, panel c shows that an investor who was well informed at time 1 (and who was a momentum trader) becomes a contrarian at time 2. The reason is that her cumulative informational advantage overwhelms her marginal informational advantage and thus she prefers to accommodate noise trading and be a contrarian.

Overall, investors who obtain “hot information” (in the context of our model, investors who have a significant marginal informational advantage) optimally choose to trade aggressively on their private information. An econometrician who analyzes the trading strategies of such heterogeneous information endowments and heterogeneous trading strategies, while preserving rationality.
investors concludes that they bet on momentum, just like Moskowitz et al. (2012) find that speculators are the ones that profit from momentum. On the other hand, investors who have a significant cumulative informational advantage (who can be seen as investors with a broad historical knowledge of the market) place more aggressive bets on public information. An econometrician who analyzes their trading strategies concludes that they are contrarians, just like Moskowitz et al. (2012) find that commercial investors, or hedgers, are contrarians. This result suggests that investors who have a large marginal advantage early on and a large cumulative advantage later optimally chose to “ride the bubble” first and bet against it near the end of the run-up. On the contrary, it is less profitable to bet on momentum near the end of the run-up.

It is not only the distinction between trading strategies that matters, but also the proportion of investors who adopt various trading strategies. The proportion of momentum traders, outlined by dark gray areas in Figures 2 and 3, move over time. In that respect, our model may alternatively be viewed as one with a time-varying representative agent. Moreover, the proportion of momentum traders at any point in time should have an impact on the dynamics of asset returns; we expect a higher proportion of momentum traders to be accompanied by momentum. We now show that this conjecture is an equilibrium implication of our model.

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Figure 3: Information Percolation and Heterogeneous Trading Strategies at \( t = 2 \)

The Figure depicts the same coefficient as in Figure 2, at time 2, i.e., \( m_2 \). The lines are the same as in Figure 2. Panel (a) is for the trader who at time 1 was the 5% less informed investor, panel (b) is for the trader who at time 1 was the average investor, and panel (c) is for the trader who at time 1 was the 95% better informed investor.

\[ \text{(a) 5\% percentile} \quad \text{(b) Average} \quad \text{(c) 95\% percentile} \]

\[ \text{Meeting intensity } \lambda \]

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\(^{20}\)We thank Sergio Rebelo por pointing out this interpretation.
4 Information Percolation and Momentum

The momentum effect presents an important challenge to theoretical explanations: if momentum is a profitable empirical anomaly easily identifiable from past returns, why does it persist? In our rational-expectations equilibrium model, we show that returns exhibit momentum, some agents trade on it, and yet momentum persists. It is therefore possible to build a rational theory in which momentum appears as a profitable anomaly. We explain below how word-of-mouth communication generates these results.

4.1 Momentum

To measure momentum, we follow Banerjee, Kaniel, and Kremer (2009) and regress next-period returns on last-period returns:

\[ E[\Delta \tilde{P}_t|\Delta \tilde{P}_{t-1}] = \frac{\text{cov}(\Delta \tilde{P}_t, \Delta \tilde{P}_{t-1})}{\text{var}(\Delta \tilde{P}_{t-1})}\Delta \tilde{P}_{t-1} \equiv \rho_t \Delta \tilde{P}_{t-1}. \]  

(11)

The coefficient \( \rho_t \), the formula of which is given in Appendix A.3, dictates whether returns exhibit momentum (\( \rho > 0 \)) or reversal (\( \rho < 0 \)).

To understand how word-of-mouth communication affects returns, we first consider an economy without social interactions and obtain the result highlighted in the proposition below, the proof of which is available in Appendix A.3.

**Proposition 3.** In the absence of word-of-mouth communication among investors, returns exhibit reversal.

Reversal is a standard result in noisy rational-expectations models; an unanticipated increase in the supply of the risky asset increases its risk premium because risk averse investors must take the other side of the market and act as market makers. An increase in the risk premium decreases the price of the asset, thereby increasing its return for the next period, hence the negative serial covariance. The magnitude of these return reversals depend on the quality of private information. In particular, with no information or with perfect information, supply shocks are anticipated and thus the serial covariance is zero. In between, the serial covariance is negative, as shown in Banerjee et al. (2009).

We now allow information to percolate through social interactions and show how it generates momentum. Our theory has three central implications: i) all investors observe momentum, ii) some investors trade profitably on it and iii) momentum is not completely eliminated by momentum trading.
4.1.1 Momentum and Momentum Traders

In this section, we show that word-of-mouth communication generates a large fraction of momentum traders and a sustained increase in average market precision over time that ultimately leads to momentum. To do so, we plot the serial correlation for the first- (the solid blue line) and the second-period returns (the dashed red line) in the top panel of Figure 4.\footnote{It is straightforward to show that the sign of the coefficient $\rho_t$ in (11) is the same as the sign of the correlation between two consecutive returns.}

For low meeting intensities ($\lambda < 0.5$), returns exhibit reversal and all investors are contrarians; this is the result we explained above. As social interactions intensify ($\lambda > 0.5$), the flow of information accelerates and the average precision of private information rises, introducing strong persistence in stock prices. The resulting persistence gradually dominates the reversal effect caused by noise trading and returns eventually exhibit momentum.

Momentum does not necessarily go hand in hand with momentum trading. In particular, for an intermediate range of intensities ($0.5 < \lambda < 1$), momentum arises even though all investors are contrarians. But, as the bottom panel of Figure 4 shows, a fraction of momentum traders emerge for a sufficiently large meeting intensity $\lambda$. As information percolation further intensifies, the magnitude of momentum ultimately decreases while more and more investors trade on it. Overall, our model predicts a hump-shaped relation between momentum and the

Figure 4: Serial Correlation in Returns and Fraction of Momentum Traders

The first and the second panel represent the serial correlation of returns and the fraction of momentum traders, respectively. Each is represented at time $t = 1$ (the solid blue line) and at time $t = 2$(the dashed red line). The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.
information diffusion speed.\footnote{Hong et al. (2000) analyze the relationship between firm size (as a proxy for speed of information diffusion) and the magnitude of the momentum. They document a hump-shaped pattern, similar to the one shown in Figure 4.}

The mechanism whereby information percolation creates momentum critically relies on a gradual increase in average market precision, as in Holden and Subrahmanyam (2002) and Cespa and Vives (2012). Our explanation for momentum is, however, distinct from theirs in that word-of-mouth communication endogenously produces this effect. Furthermore, a gradual increase in precision alone does not lead to momentum trading (as we will show in Section 4.2.1). For momentum trading to obtain, another key feature is needed—the heterogeneity in information endowment across agents. In our theory for momentum, social interactions simultaneously generate an increase in information precision and heterogeneous information endowments.

### 4.1.2 Market Learning and Momentum

Momentum arises in our model although investors can easily identify it from past prices and potentially eliminate it through trading. We argue here that the simple fact that investors can detect a trend from prices plays a key role in generating momentum. To see this, notice that investors face an adverse selection problem: a price move may reflect fundamental information (informed trading) or idiosyncratic noise (noise trading). Information percolation alleviates this problem by making it easier for investors to filter fundamental information from prices. Suppose that we know the fundamental value $\tilde{U}$ of the asset (say it is $\tilde{U} = 1$); we can then compute how prices converge to the fundamental on average

$$E[\tilde{P}_t|\tilde{U}] = \left(1 - \frac{H}{K_t}\right)\tilde{U} \equiv T_t\tilde{U}.$$ 

The coefficient $T$ reflects the fundamental trend in prices. As the meeting intensity goes to infinity, $T = 1$ and prices match the fundamental on average. We plot the fundamental trend $T$ in Figure 5.

For low meeting intensities (the black solid line), prices slowly converge to the fundamental and thus the fundamental trend in prices is weak. As social interactions intensify, the flow of information through prices accelerates and the fundamental trend gets more and more pronounced. An accelerated price convergence—equivalently, a more and more pronounced trend—simultaneously generates momentum and makes it easier for investors to identify it.

Everyone observes momentum, but not everyone decides to trade on it. By accelerating price convergence, information percolation alleviates investors’ adverse selection problem and therefore decreases the risk premium for holding the asset over time. As a result, some investors
Figure 5: Fundamental Trend in Prices

The Figure depicts the fundamental trend in prices over time. The fundamental trend is plotted for a meeting intensity of $\lambda = 0$ (the black solid line), $\lambda = 1$ (the blue dotted line), and $\lambda = 2$ (the red dashed line). The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

...optimally decide to accommodate noise traders’ demand, thus pursuing a contrarian strategy. Overall, information from prices and word-of-mouth communication operate simultaneously to generate momentum and heterogeneous trading strategies.

This last point helps distinguish our model from behavioral explanations, in particular those of Hong and Stein (1999) and Daniel et al. (1998). Our explanation helps reconcile both theories: to produce momentum, we need prices not to reflect information instantly—the under-reaction explanation of Hong and Stein (1999)—but this information must necessarily be accelerated (through word-of-mouth communication)—in the spirit of the continuing over-reaction explanation of Daniel et al. (1998).23 But our explanation also relies on rational learning from prices: unlike Hong and Stein (1999) who preclude learning from prices and exogenously postulate momentum traders, in our case investors rationally process information from prices and a fraction of them decide endogenously to trade on momentum.24

23Note that investors do not over-react to information in our setup.
24Our rational explanation is also distinct from that of Makarov and Rytchkov (2009) and Albuquerque and Miao (2014). Using a rational-expectations equilibrium model, Makarov and Rytchkov (2009) show that the slow diffusion of information may produce momentum, an under-reaction phenomenon. In our model, instead, we need that information be accelerated to produce momentum. Albuquerque and Miao (2014) construct a dynamic model in which informed investors receive “advanced” information about the fundamental. Prices adjust to reflect this information and keep moving in the same direction as this information is further corroborated by the arrival of public information. In their model, information is instantly incorporated into prices and momentum arises as investors under-react to this information. In our model, prices do not
The line represents the profits on a naive momentum strategy, performed by an econometrician, as a function of the meeting intensity $\lambda$. The econometrician only observe prices and buys one unit of the asset if prices went up or sells one unit of the asset if prices went down. The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

4.1.3 Momentum Profits

Any investor who has access to past prices can make profits on momentum—no further knowledge is needed. We take now the view of an econometrician (who has no information other than past and current prices) and study whether a simple momentum strategy is profitable. We compute the expected cumulative profits of a naive momentum strategy which starts at time 1 and consists in buying one unit of the stock at time $t$ if prices went up ($\Delta \tilde{P}_t > 0$) or selling one unit of the stock if prices went down ($\Delta \tilde{P}_t < 0$):

\[
\text{Momentum Profits} = \mathbb{E} \left[ \sum_{t=1}^{3} \left( 1_{\Delta \tilde{P}_t > 0} - 1_{\Delta \tilde{P}_t < 0} \right) \Delta \tilde{P}_{t+1} \right]
\]

We plot econometrician’s expected profits in Figure 6. For low meeting intensities, the econometrician’s strategy performs poorly: returns exhibit reversal and a naive momentum strategy is suboptimal. As information percolation accelerates, the profits become positive.
and their magnitude strongly relates with the magnitude of momentum (as depicted in Figure 4). Any agent in our economy can therefore make profits on momentum.\footnote{In our model, agents choose optimal portfolios that are more sophisticated than a naive momentum strategy (regardless of their information endowment), because they know the structure of the economy as described in Theorem 1. They are therefore free to chose whether they perform momentum or contrarian trading. The point that we want to make here is that, indeed, momentum appears to the econometrician as a profitable empirical anomaly.}

4.2 Discussion of Alternative Models

Our modeling strategy raises at least two questions that we address in this Section. First, how is our model different from an alternative one in which the average market precision increases exogenously and generates momentum (Holden and Subrahmanyam, 2002)? Second, is momentum in our model an artefact of the single liquidation date?

We show that word-of-mouth communication not only generates momentum, but also generates heterogeneity in trading strategies across agents. In a model in which the precision of information increases exogenously, traders are only contrarians. Such a model does not generate momentum trading and therefore cannot explain why momentum is not entirely eliminated by momentum trading. We address the concern related to the finite horizon of our model by building a setup in which the asset pays an infinite stream of dividends instead of a single liquidating dividend. Even in this case, information percolation generates or amplifies momentum. We conclude that the intuition presented so far is not entirely driven by the fact that the world is run once and gradually comes to an end.

4.2.1 Alternative Model with Exogenous Increase in Precision

We assume away word-of-mouth communication among investors and propose a model in which investors’ signal precision $S_t$ arbitrarily improves over time ($S_t > S_{t-1}$). We present the resulting equilibrium in Theorem 2.

**Theorem 2 (An economy with exogenous increase in information precision).** There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price, $\tilde{P}_t$, and individual asset demands, $\tilde{D}_t$, for $t = 0, \ldots, 3$, are given by:

$$\tilde{P}_t = \frac{K_t - H}{K_t}\tilde{U} - \sum_{j=0}^{t} \frac{1 + \gamma^2 S_j \Phi}{\gamma K_t} \tilde{X}_j$$
and

\[ \tilde{D}_t^i = \gamma \sum_{j=0}^{t} S_j \left( \tilde{Z}_j^i - \tilde{Q}_j \right) \]

where

\[ \tilde{Q}_t \equiv \tilde{U} - \frac{1}{\gamma S_t} \tilde{X}_t \]

\[ K_t \equiv \text{Var}^{-1} \left[ \tilde{U} \right] = H + \sum_{j=0}^{t} S_j + \sum_{j=0}^{t} \gamma^2 \Phi S_j^2 \]

\[ \tilde{\mu}_t^i \equiv \mathbb{E} \left[ \tilde{U} \bigg| \tilde{F}_t^i \right] = \frac{1}{K_t} \sum_{j=0}^{t} \left[ S_j \tilde{Z}_j^i + \gamma^2 S_j^2 \Phi \tilde{Q}_j \right] \]

The optimal trading strategy of the individual investor \( i \) at time \( t = 1, 2, 3 \) is given by

\[ \Delta \tilde{D}_t^i \equiv \tilde{D}_t^i - \tilde{D}_{t-1}^i = \gamma S_t \left( \tilde{Z}_t^i - \tilde{Q}_t \right) \]

**Proof.** See Brennan and Cao (1997).

Applying the measure of momentum of Banerjee et al. (2009) to this particular setup, we obtain the following result.

**Proposition 4.** In an economy with exogenous increase in information precision, momentum arises at time \( t = 1 \) if and only if agent’s precision shows sufficient improvement over time.

**Proof.** Substituting \( S\Omega_t \equiv S_t \) in (45) and simplifying shows that returns exhibit momentum at time \( t = 1 \) if

\[ S_1 - S_0 > \frac{H}{1 + \gamma^2 \Phi S_0} > 0. \]

The right-hand side is strictly positive. \( \square \)

To generate momentum, the necessary improvement in agents’ precision is increasing in the precision \( H \) of the fundamental and risk aversion \( 1/\gamma \) and decreasing in the precision of the supply increments \( \Phi \). Furthermore, if agents’ past precision of private information is large (\( S_0 \) large), an even larger precision of private information today (\( S_1 \)) is needed to produce momentum. To illustrate this, suppose agent’s precision increases geometrically over time, i.e., \( S_t = a^t \). Using our calibration, we need at least \( a \approx 2 \) to generate momentum—the precision of the signal at time \( t = 1 \) must be at least twice that of the signal at time \( t = 0 \).
This observation allows us to highlight the first contribution of our model with respect to a model with exogenous increase in average precision: while a substantial improvement in agent’s precision may be difficult to justify exogenously, this phenomenon naturally arises when investors interact through word-of-mouth communication. Our model therefore provides a micro-foundation for this gradual increase in agents’ precision.

The second contribution of our model becomes immediately apparent when considering agents’ trading strategy. Proposition 5, directly related to Proposition 4, makes this point.

**Proposition 5.** In an economy with exogenous increase in information precision, all investors implement contrarian trading strategies at time \( t = 1 \).

**Proof.** In this setting, agents have no informational advantage \((A_i^t ≡ a_i^t ≡ 0)\), since they all have the same precision. As a result, the trading measure of section 4 simplifies to

\[
\text{cov}(\Delta \tilde{D}_i^t, \Delta \tilde{P}_i^t) ≡ -\frac{1 + \gamma^2 S_1 \Phi}{\gamma \Phi K_0} < 0
\]

This expression is negative for all admissible parameter values. □

An exogenous increase in average market precision can generate momentum with contrarian investors only. But what makes momentum a puzzle is precisely that some agents trade on it and yet that it persists. An exogenous increase in average precision therefore cannot provide a satisfactory answer to the momentum puzzle.

**4.2.2 A Stationary Model**

Our theory for momentum so far relies on an economy with a single liquidation date. In this section, we show that our results extend to a fully dynamic setup. In particular, we build a stationary version of the model and show that, even in this case, information percolation generates or amplifies momentum.

We present a simplified version of the model and relegate all technical details in Appendix A.4. We consider an economy that goes on forever and in which one risky asset (stock) pays a stochastic dividend \( D_t \) per share. As in the finite version of the model, new liquidity traders are assumed to enter the market in every trading session. Too keep things simple, let us assume that the dividend process \( D_t \) and the supply process \( X_t \) follow random walks. We will discuss more general processes at the end of this Section.

\[
D_t = D_{t-1} + \varepsilon^d_t
\]

\[
X_t = X_{t-1} + \varepsilon^x_t
\]
All investors observe the past and current realizations of dividends and of the stock prices. Additionally, each investor observes a signal about the dividend innovation 3-steps ahead:

\[ \tilde{z}_t^i = \varepsilon_{t+3}^d + \bar{\epsilon}_t^i \]

As in the baseline model, investors meet and share private information over time. A fundamental difference, however, is that investors do not talk about a single liquidation value, but about several dividends revealed at different times in the future. That is, not only do they share information about the dividend 3-steps ahead, but they also share information about the dividend 2-steps ahead, and so on.\(^{26}\)

Unlike the baseline model, we consider an overlapping generation of agents, as in Bacchetta and Wincoop (2006), Watanabe (2008), Banerjee (2010), and Andrei (2013). This assumption considerably simplifies the analysis by ruling out dynamic hedging demands.\(^{27}\)

The solution method, which follows Andrei (2013), proceeds by specifying an equilibrium price that is a linear function of model innovations:

\[ P_t = \alpha D_t + \beta X_{t-3} + (a_3 a_2 a_1)\varepsilon_t^d + (b_3 b_2 b_1)\varepsilon_t^x \]

where \(\varepsilon_t^d \equiv (\varepsilon_{t+1}^d \varepsilon_{t+2}^d \varepsilon_{t+3}^d)^\top\) are the 3 future unobservable dividend innovations and \(\varepsilon_t^x \equiv (\varepsilon_{t-2}^x \varepsilon_{t-1}^x \varepsilon_t^x)^\top\) are the last 3 supply innovations. The main difference with our static baseline model is that this equilibrium functional form is now stationary. That is, the coefficients \(\alpha, \beta, a,\) and \(b\) do not change over time whereas in Theorem 1 the price coefficients change as the economy approaches the finite end-point.

We now show that information percolation generates momentum, even though prices are stationary. The random walk specification (46) - (47) helps us clearly isolate the effect of information percolation on prices. Indeed, when investors do not have private information, this specification directly implies that stock returns are serially uncorrelated. The solid blue line in Figure 7 then shows that, once investors receive private information, stock returns exhibit momentum, which information percolation further amplifies. The same intuition applies when the dividend and supply processes are not random walks. Because these processes are now mean-reverting, stock returns exhibit reversals in most of the

\(^{26}\)Note that the model can be extended to a general case in which investors receive information about the dividend \(T\)-steps ahead at the expense of analytical complexity and without altering the main intuition presented here.

\(^{27}\)In the infinite-horizon case the portfolio maximization problem is substantially more complicated. The fixed point problem cannot be reduced to a finite dimensional one, but Bacchetta and Wincoop (2006) and Andrei (2013) show how to approximate the problem to a desired accuracy level by truncating the state space. The (numerical) results for the infinite horizon model are very close to those obtained in the overlapping generations model. See also Albuquerque and Miao (2014).
Figure 7: Serial Correlation of Returns in the Stationary Model
The figure depicts the serial correlation of returns, $\text{corr}(P_{t+1} - P_t, P_{t+2} - P_{t+1})$, for different levels of the meeting intensity $\lambda$. There are two cases: (i) the dividend and supply processes are random walks (solid blue line) and (ii) the dividend and supply processes are mean-reverting with AR(1) parameter 0.9. The calibration for the rest of the parameters ensures the existence of an equilibrium in the stationary model: $R = 1.1$, $H = 1$, $S = 10$, $\Phi = 1/100$, and $\gamma = 3$, although most of the calibrations we have tried give the same qualitative results.

5 Information Percolation and Reversal
Social interactions are natural propagators of rumors. In this section, we show that a “rumor” can generate a phase of price over-shooting followed by a phase of price correction. Under certain conditions, this convergence pattern can jointly produce short-term momentum and long-term reversal.

We introduce a “rumor” in the setup of Section 2. We follow Peterson and Gist (1951) and define a rumor as “an unverified account or explanation of events circulating from person to person and pertaining to an object, event, or issue in public concern.” To introduce this aspect in our model, we consider a small modification of the previous setup and assume that agents receive signals of the form:

$$\tilde{z}_t^i = \tilde{U} + \tilde{V} + \tilde{\epsilon}_t^i$$

where $\tilde{V}$ is normally distributed with zero mean and precision $\nu$.

The common noise, $\tilde{V}$, satisfies two important properties of a rumor (as defined above): (i) it circulates from person to person and (ii) it is unverifiable. The first property arises as we
allow agents to interact through word-of-mouth communication—private signals now contain a rumor that is circulated from one agent to another. The second property results from the signal specification in (14): on average, private signals only reveal the sum of the fundamental value and the rumor \((\bar{U} + \bar{V})\). As a result, agents cannot distinguish fundamental information from the rumor, either using past prices or their private signals. The rumor is therefore unverifiable.

Rumors do not last forever, but eventually subside. To incorporate this aspect, we assume that each agent receives a signal at time \(t = 3\) that is centered on the true value of the fundamental:

\[
\tilde{Z}_3^i = \bar{U} + \bar{e}_3^i.
\]

Using this signal, agents can back out the content of the rumor (on average) at time \(t = 3\) and the rumor subsides. Overall, agents are aware of the rumor, but cannot learn about its content until time \(t = 3\).

In the presence of a rumor, asset prices and investor asset demands do not have a closed-form solution. Theorem 3 describes a system of recursive equations for the equilibrium price coefficients. We provide the proof of Theorem 3 and we solve this system of equations through an efficient numerical scheme that we describe in Appendix A.5.

**Theorem 3.** In the presence of a rumor, the price \(\tilde{P}\) is informationally equivalent to

\[
\tilde{Q}_t = \bar{U} + \frac{\Lambda_t}{rS} \bar{V} - \frac{1}{rS} \tilde{X}_t
\]

where the equilibrium coefficients \(\Omega\) and \(\Lambda\) solve a fixed-point problem given by a system of recursive equations:

\[
\Omega_t = \frac{1}{rS} \sum_{j=0}^{t} \bar{\theta}_j - \sum_{j=0}^{t-1} \Omega_j \tag{15}
\]

\[
\Lambda_t = \sum_{j=0}^{t} \bar{\theta}_j - \sum_{j=0}^{t-1} \Lambda_j
\]

in which \(\bar{\theta}\) denotes the average coefficients in front of agents’ private signals in their optimal demand.

The rumor has two important effects on prices, the first of which is immediately apparent when looking at the coefficient \(\Omega\). We plot this coefficient in Figure 8, both when signals contain a rumor (panel (b) with \(\nu = 3\)) and when they do not (panel (a)).

When signals do not contain a rumor, the coefficient \(\Omega\) represents the average number of incremental signals. Panel (a) shows that this average rises as time passes by, all the more so
when social interactions intensify. When signals contain a rumor (panel (b)), the coefficient $\Omega$ also keeps increasing initially, but much less so as compared to panel (a). Intuitively, agents know they possess information of lower quality due to the presence of the rumor. Therefore, they apply a discount on the actual number of signals they have—the coefficient $\Omega$ now represents a discounted average of incremental signals.

At time $t = 2$, the discounted average $\Omega$ declines: agents anticipate that they will get better information at time $t = 3$ and apply a stronger discount on their number of signals. At time $t = 3$, the discounted average number of signals reaches zero for $\lambda = 3$: when agents have collected a vast number of signals, they can accurately forecast $\tilde{U} + \tilde{V}$. Hence, when they get the signal that is centered on the fundamental, they basically ignore all the signals they have accumulated. Overall, the rumor induces agents to interpret their information with some caution.

A second important effect of the rumor relates to price convergence; it is precisely this effect that allows the rumor to produce reversals in stock returns. In particular, the rumor causes the price to “over-shoot” its fundamental value. To show this, we proceed as in Section 4.1.2: suppose we know both the fundamental value of the asset (say $\tilde{U} = 1$) and the content of the rumor (say $V = 1/3$). We can then compute how the price converges to the fundamental on average. We do so in Figure 9.

Ruling out social interactions (the solid black line), prices converge slowly towards the
The Figure depicts the average price convergence over time in the presence of a rumor. The price convergence is plotted for a meeting intensity of $\lambda = 0$ (the black solid line), $\lambda = 1$ (the blue dotted line), and $\lambda = 2$ (the red dashed line). The calibration is $H = S = \Phi = 1$, $\gamma = \frac{1}{3}$, and $\nu = 6$.

fundamental value of the stock. This convergence pattern is similar to the one described in Section 4.1.2. When we introduce social interactions, this pattern changes: not only do prices converge faster toward the fundamental of the stock, but, most important, they *over-shoot* the fundamental value of the stock. Social interactions and the rumor jointly produce this *hump-shaped* pattern. To see this, notice that social interactions initially accelerate price convergence towards the sum of the fundamental and the rumor causing the price to over-shoot the fundamental. At time $t = 3$, agents receive information of better quality and the price corrects—the rumor subsides.

We now investigate how this convergence pattern relates to the serial correlation of stock returns. Intuitively, the first phase of price “over-shooting” generates short-term momentum and the second phase of price correction generates long-term reversal. To show this, we proceed as in Section 4.1 and plot returns’ serial correlation in Figure 10.

When the rumor is fairly precise (panel (a)), returns mostly exhibit momentum: despite the presence of the rumor, agents’ precision rises over time, generating momentum. As the precision of the rumor decreases (panel (b)), agents discount their actual number of signals more strongly. As a result, agents progressively cut back their positions—they adjust their trades to reflect that their information is of lower quality. While these portfolio adjustments do not prevent returns to exhibit momentum in the first-period, they induce reversal in the
second period as the price gradually corrects. Finally, when the rumor’s precision is low (panel (c)), agents become extremely cautious about their information and the improvement in their precision is not sufficient to generate momentum. As a result, returns mostly exhibit reversal. Overall, our model can jointly generate short-term momentum—consistent with the empirical finding of Jegadeesh and Titman (1993)—and long-term reversal—consistent with the over-reaction phenomenon of De Bondt and Thaler (1985).

6 Conclusion

We show that the diffusion of information in financial markets produces momentum in asset returns. When investors accumulate information through word-of-mouth communication, the price convergence towards the fundamental is accelerated, producing a trend. All investors observe this trend and some decide to bet on it, but, due to the rich heterogeneity in investment strategies generated by word-of-mouth communication, momentum is not completely eliminated. Momentum thus appears to the econometrician as a profitable empirical anomaly. Furthermore, we show that word-of-mouth can spread rumors and generate price overshooting. The momentum phase is followed by a significant reversal, in line with the widely documented
under- and over-reaction patterns in stock returns.

A legitimate question is what empirical exercise would validate our model and distinguish it from other theoretical models generating the same results. We believe that natural experiments capturing an exogenous increase or decrease in the intensity of word-of-mouth communication could make a worthwhile empirical point. For example, Shiller (2000) relates the obvious increase in the word-of-mouth communication intensity once the telephone became effective during the 1920s with the steady increase of volatility during the same period. Another option is to study the consequences of the Regulation Fair Disclosure, promulgated by the U.S. Securities and Exchange Commission in August 2000. This regulation forbids firms and their insiders to provide information to some investors (often large institutional investors). Hence, after August 2000 there should be less information propagated through the word-of-mouth communication channel.

By putting forward a social feature of humans (interpersonal communication), our research can be viewed as complementary to the behavioral finance literature. Even though we explain momentum without assuming any individual behavioral bias, we believe that adding behavioral biases as in Daniel et al. (1998) or Barberis et al. (1998) would amplify the effects analyzed in this paper. Other questions are worthwhile investigating, such as extending the setup to multiple assets, where information percolation could generate rich dynamics of the conditional correlation among assets. It is also interesting to study precisely the mechanism of information transmission and find conditions under which investors find it beneficial to tell the truth (Stein, 2008).
References


A Appendix

A.1 Information Percolation

A.1.1 Distribution of Incremental Signals: Closed-Form Solution

To obtain the closed-form solution for the distribution \( \pi \) of incremental signals, we first derive the equation for its dynamics.

**Lemma 1.** The probability density function \( \pi \) over the additional number of signals collected by each agent satisfies

\[
\frac{d}{dt} \pi_t(n) = -\lambda \pi_t(n) + \lambda (\pi_t * \mu_t)(n), \quad \pi_0 = \delta_{n=1}.
\]

**Proof.** We compute the distribution of new signals that have been gathered between time 0 and time \( T \). Denote by \( X_i \), the number of new signals gathered if a meeting occurs at time \( t_i \) and observe that it is distributed as

\[
X_i \sim \mu(t, \cdot)
\]

where the distribution \( \mu(t, x) \) satisfies the differential equation in (1). Furthermore, the number \( N(T) \) of meetings that took place between time 0 and \( T \) is a Poisson counter with intensity \( \lambda \); accordingly, the total number \( Y_T \) of new signals gathered between time 0 and \( T \) is given by \( \sum_{i=1}^{N(T)} X_i \). We now characterize its distribution. First, observe that \( Y_T \), conditional on the set of times \( \{0 \leq t_1 \leq t_2 \leq \ldots \leq t_{N(T)} \leq T\} \) at which a meeting occurs (up to time \( T \)) and the total number of meetings \( N(T) \) (that is, conditioning on the whole trajectory \( A^N_T \) of the Poisson process), is distributed as

\[
Y_T|A^N_T \sim \int_{\mathbb{R}} \mu(Y_{t_{N-1}} - Y_{t_{N-1}}, t_{N}) \, d\mu(Y_{t_{N-1}} - Y_{t_{N-2}}, t_{N-1}) \ldots d\mu(Y_{t_1} - 0, t_1)
\]

Second, observe that the distribution \( \mu(X_{t_i}, t_i) \) of increment can be expressed as a translation \( T \) of the type measure \( \mu \) and the increment \( x \). Hence, the distribution in (17) may be written as

\[
Y_T|N(T) \sim \Gamma_{i=1}^{N(T)} \mu_{t_i}
\]

where, for any probability measures \( \alpha_1, \ldots, \alpha_k \), we write \( \Gamma_{i=1}^{k} = \alpha_1 * \alpha_2 * \ldots * \alpha_k \).

Now, observe that each \( t_i \) in the sequence of meetings \( \{0 \leq t_1 \leq t_2 \leq \ldots \leq t_{N(T)} \leq T\} \) conditional on \( N(T) \) is uniformly distributed over \( T \); accordingly, we have that

\[
Y_T|N(T) \sim \Gamma_{i=1}^{N(T)} \frac{1}{T} \int_0^T \mu_{t_i} dt_i = \left( \frac{1}{T^{N(T)}} \int_0^T \mu_s ds \right)^{*N(T)}
\]

where \( *n \) denotes the \( n \)-fold convolution.

Finally, since \( N(T) \) is a Poisson(\( \lambda \)) counter, we have

\[
Y_T \sim \sum_{k=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^k}{k!} \frac{1}{T^k} \left( \int_0^T \mu_s ds \right)^{*k} = \sum_{k=0}^{\infty} e^{-\lambda T} \frac{\lambda^k}{k!} \left( \int_0^T \mu_s ds \right)^{*k}.
\]
Using the fact that, by Taylor expansion, \( e^x \) is equivalently written as \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \), we can write \( \pi_T = e^{\lambda (T_0 + \mu_s ds - T)} \).

Differentiating this expression, we obtain (16). ■

A.1.2 Proof of Proposition 1

To obtain the closed form, it is convenient to think of individual signals received at times \( t = 0, 1, 2 \) separately. That is, even though all are signals about the same fundamental, we will treat them separately. Assume that signals received at time \( t = 0, 1, 2 \) are of type \( s_0, s_1, \) and \( s_2 \) respectively.

Let us fix some notation:

0. \( n_0 \equiv \) the total number of type \( s_0 \) signals received between time 0 and \( t \).
1. \( n_1 \equiv \) the total number of type \( s_1 \) signals received between time 1 and \( t \).
2. \( n_2 \equiv \) the total number of type \( s_2 \) signals received between time 2 and \( t \).

and:

0. \( m_0 \equiv \) the incremental number of type \( s_0 \) signals received between the last trading date and \( t \).
1. \( m_1 \equiv \) the incremental number of type \( s_1 \) signals received between the last trading date and \( t \).
2. \( m_2 \equiv \) the incremental number of type \( s_2 \) signals received between the last trading date and \( t \).

Let us focus first on the total number of signals, \( n_0, n_1, \) and \( n_2 \). From time 0 to 1, the distribution of \( n_0 \) has the support \([1, \infty)\). Recursive computations show that this distribution is

\[
\mu_{\text{total}, n_0} = e^{-n_0 \lambda \tau} \left( e^{\lambda \tau} - 1 \right)^{n_0 - 1}
\]

where \( 0 \leq \tau \leq 1 \).

From time 1 to 2, the distribution of \( \{n_0, n_1\} \) has the support \([1, \infty) \times [1, \infty)\). Further recursive computations show that this distribution is

\[
\mu_{\text{total}, n_0, n_1} = \left\{ \begin{array}{ll}
\left( n_0 \div n_1 \right) e^{-n_0 \lambda - n_1 \lambda \tau} \left( e^{\lambda \tau} - 1 \right)^{n_0 - n_1} & \text{if } n_0 \geq n_1 \\
0, & \text{otherwise}
\end{array} \right.
\]

From time 2 to 3, the distribution of \( \{n_0, n_1, n_2\} \) has the support \([1, \infty) \times [1, \infty) \times [1, \infty)\). Further recursive computations show that this distribution is

\[
\mu_{\text{total}, n_0, n_1, n_2} = \left\{ \begin{array}{ll}
\left( n_0 \div n_1 \div n_2 \right) e^{-n_0 \lambda - n_1 \lambda - n_2 \lambda \tau} \left( e^{\lambda \tau} - 1 \right)^{n_0 - n_1 - n_2} & \text{if } n_0 \geq n_1 \geq n_2 \\
0, & \text{otherwise}
\end{array} \right.
\]

Focus now on the distribution of increments, \( m_0, m_1, \) and \( m_2 \). From time 0 to 1, the distribution of \( m_0 \) has the support \([0, \infty)\). Recursive computations show that this distribution is

\[
\mu_{\text{incr}, n_0} = e^{(-m_0 + 1) \lambda \tau} \left( e^{\lambda \tau} - 1 \right)^{m_0}
\]
To obtain the distribution of incremental signals numerically, we proceed through discrete Fourier computations show that this distribution is
\[
\mu\text{incr}_{m_0, m_1} = \begin{cases}
\frac{(m_0 - 1)}{(m_1 - 1)}e^{-m_0\lambda - (m_1 + 1)\lambda r} \left( e^{\lambda} - 1 \right)^{m_0 - m_1} \left( e^{\lambda r} - 1 \right)^{m_1}, & \text{if } m_0 \geq m_1 \\
0, & \text{otherwise}
\end{cases}
\]
(19)

From time 2 to 3, the distribution of \( \{m_0, m_1, m_2\} \) has the support \([0, \infty) \times [0, \infty) \times [0, \infty) \). Further recursive computations show that this distribution is
\[
\mu\text{incr}_{m_0, m_1, m_2} = \begin{cases}
\frac{(m_0 - 1)}{(m_1 - 1)}(m_2 - 1)e^{-m_0\lambda - m_1\lambda - (m_2 + 1)\lambda r} \left( e^{\lambda} - 1 \right)^{m_0 - m_2} \left( e^{\lambda r} - 1 \right)^{m_2}, & \text{if } m_0 \geq m_1 \geq m_2 \\
0, & \text{otherwise}
\end{cases}
\]
(20)

We can now group the signals of the same type, since they are all informative about the same fundamental. From time 0 to 1, signals are only of type \( s_0 \) and thus the probability density function over the additional number of signals follows from (18) with \( \tau = 1 \), with \( m_0 = n - 1 \):
\[
\pi_1(n) = e^{-n\lambda} \left( e^{\lambda} - 1 \right)^{n - 1}
\]

From time 1 to 2, given a number \( n \) of additional signals one has to find all the combinations of \( m_0 \) and \( m_1 \) for which \( m_0 + m_1 = n - 1 \), and then make the sum of all the corresponding terms in (19) with \( \tau = 1 \):
\[
\pi_2(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \left[ \binom{i}{j - 1} \binom{j}{1} \right] e^{-i\lambda - (j + 1)\lambda} \left( e^{\lambda} - 1 \right)^{i}
\]

From time 2 to 3, given a number \( n \) of additional signals one has to find all the combinations of \( m_0, m_1, \) and \( m_2 \) for which \( m_0 + m_1 + m_2 = n - 1 \), and then make the sum of all the corresponding terms in (20) with \( \tau = 1 \):
\[
\pi_3(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \left[ \binom{i}{j - 1} \binom{j}{k - 1} \right] e^{-(i + j + k + n - 1)\lambda} \left( e^{\lambda} - 1 \right)^{i}
\]

### A.1.3 Numerical Approach: Fourier Transform

To obtain the distribution of incremental signals numerically, we proceed through discrete Fourier transforms. Denote the Fourier transform of \( \mu \) and \( \pi \) by \( \hat{\mu}(z) := \int_{\mathbb{R}} e^{izn} d\mu(n) \) and \( \hat{\pi}(z) := \int_{\mathbb{R}} e^{izn} d\pi(n) \), respectively, where \( i = \sqrt{-1} \) and \( z \in \mathbb{R} \). As in Duffie and Manso (2007), \( \hat{\mu} \) is given in closed form:
\[
\hat{\mu}(z) = \frac{\hat{\mu}_0(z)}{e^{M(1 - \hat{\mu}_0(z))} + \hat{\mu}_0(z)}
\]
where \( \hat{\mu}_0(z) = e^{iz} \) since \( \mu_0(n) \) is a Dirac mass at 1.

To obtain \( \hat{\pi} \), we integrate (16) and get the following equation
\[
\frac{d}{dt} \hat{\pi}(z) = -\lambda \hat{\pi}(z) + \lambda \hat{\mu}(z) \hat{\pi}(z),
\]
the solution of which is also available in closed form:

\[ \hat{\pi}_t(z) = \frac{\hat{\pi}_0(z)e^{i\pi}}{e^{M(\hat{\mu}_0(z) - 1)} - \hat{\mu}_0(z)} \]

where \( \hat{\pi}_0(z) = e^{iz} \) since \( \pi_0(n) \) is a Dirac mass at 1.

We can now recover \( \pi \) numerically by using the inverse Fourier formula. To do so, notice that both distributions \( \mu(n) \) and \( \pi(n) \) are so-called lattice distributions for which every possible realization of \( n \) can be represented as \( a + bk \) where \( k \) only takes integral values—in our case, \( n \in \mathbb{N}, a = \frac{N}{2} \), and \( b = \frac{1}{2} \). For this class of discrete distributions, the inverse Fourier formula writes

\[
P[X = x_k] = \frac{b}{2\pi} \int_{-\pi/b}^{\pi/b} e^{-ixx_k} \hat{\mu}_t(z) \, dz.
\]

To compute the integral in (21), we use fast Fourier Transform: we rewrite (21) as

\[
P[X = x_k] = \frac{b}{\pi} \int_{0}^{\pi/b} e^{-ixx_k} \hat{\mu}_t(z) \, dz.
\]

We then use the trapezoid rule rule to discretize the integral:

\[
P[X = x_k] \approx \frac{b}{\pi} \frac{e^{-\frac{ix\pi}{2}}}{2} \hat{\mu}_t \left( \frac{\pi}{b} \right) \Delta z + \frac{b}{\pi} \sum_{j=0}^{M-1} \delta_j e^{-ij\Delta x x_k} \hat{\mu}_t (j \Delta z) \Delta z
\]

with \( \delta_j = \frac{1}{2} \) if \( j = 1 \) and \( \delta_j = 1 \) otherwise. We need to choose \( \Delta z \) such that the upper integration bound \( \frac{\pi}{b} = 2\pi \) is reached by \( z \); accordingly, we set \( \Delta z \equiv \frac{2\pi}{M} \). Furthermore, the grid points for \( x_k \) are \( x_k = -d + \lambda k, \ j \in \mathbb{N} \) and the fast Fourier transform method imposes that \( \lambda \Delta z = \frac{2\pi}{M} \), i.e., \( \lambda = 1 \). Since \( x_k \in \mathbb{N} \), we must have that \( d = 0 \). As a result, \( P[X = x_k] \) takes the form of a discrete Fourier transform:

\[
P[X = x_k] \approx \frac{1}{M} \sum_{j=0}^{M-1} g_j e^{-j\frac{2\pi}{M}i}
\]

with

\[
g_j = \delta_j \frac{M}{2\pi} \hat{\mu}_t \left( \frac{2\pi}{M} \right) \frac{2\pi}{M}.
\]

Finally, since each agent gets an additional signal every period, we must use the discrete Fourier transform sequentially: to derive the distribution \( \pi \) at time \( t+1 \), we use the numerical expression for \( \mu_t(n) \) that we computed at time \( t \) and compute \( \hat{\mu}_0(z) = \sum_{k=1}^{N} e^{izk} f_0(k) \) where

\[
f_0(n) = \begin{cases} 
\mu_t(n-1) & \text{if } n \geq t + 1 \\
0 & \text{otherwise}
\end{cases}
\]

We then substitute \( \hat{\mu}_0(n) \) into \( \hat{\pi}_t(n) \big|_{t=1} \) and apply the inverse Fourier formula, which yields \( \pi_{t+1} \).
A.2 Proof of Theorem 1

We provide the proof for a two trading session economy, that is, we eliminate for ease of exposition dates \( t = 2 \) and \( t = 3 \). Once the equilibrium quantities are written in a recursive form, as in Brennan and Cao (1997), or in He and Wang (1995), it is straightforward to derive the full recursive equilibrium solution.

The model is solved backwards, starting from date 1 and then going back to date 0. First, conjecture that prices in period 0 and period 1 are

\[
\tilde{P}_0 = \beta_0 \tilde{U} - \alpha_{0,0} \tilde{X}_0 \\
\tilde{P}_1 = \beta_1 \tilde{U} - \alpha_{1,0} \tilde{X}_0 - \alpha_{1,1} \tilde{X}_1
\]  

(22, 23)

Consider the normalized price signal in period zero (which is informationally equivalent to \( \tilde{P}_0 \)):

\[
\tilde{Q}_0 = \frac{1}{\beta_0} \tilde{P}_0 = \tilde{U} - \frac{\alpha_{0,0}}{\beta_0} \tilde{X}_0
\]  

(24)

Replace \( \tilde{X}_0 \) from (24) into (23) to obtain

\[
\tilde{P}_1 = \varphi_1 \tilde{U} + \xi_1 \tilde{Q}_0 - \alpha_{1,1} \tilde{X}_1
\]  

(25)

where \( \varphi_1 = \beta_1 - \alpha_{1,0} \frac{\beta_0}{\alpha_{0,0}} \) and \( \xi_1 = \alpha_{1,0} \frac{\beta_0}{\alpha_{0,0}} \). These coefficients are to be determined in equilibrium. We normalize the price signal in period \( t = 1 \) and obtain \( \tilde{Q}_1 \):

\[
\tilde{Q}_1 = \frac{1}{\varphi_1} \left( \tilde{P}_1 - \xi_1 \tilde{Q}_0 \right) = \tilde{U} - \frac{\alpha_{1,1}}{\varphi_1} \tilde{X}_1
\]

Observing \( \{\tilde{Q}_0, \tilde{Q}_1\} \) is equivalent with observing \( \{\tilde{P}_0, \tilde{P}_1\} \). We conjecture the following relationships (see Admati 1985):

\[
\frac{\alpha_{0,0}}{\beta_0} = \frac{1}{\gamma S \Omega_0} \\
\frac{\alpha_{1,1}}{\varphi_1} = \frac{1}{\gamma S \Omega_1}
\]  

(26, 27)

where \( \Omega_t \) is the cross-sectional average of the number of additional signals at time \( t = 0, 1 \), \( \Omega_t = \sum_{\omega \in \Omega} \omega \pi_\ell(\omega) \). In our setup, \( \Omega_0 = 1 \forall \lambda, \Omega_1 = 1 \) if \( \lambda = 0 \), and \( \Omega_1 > 1 \) if \( \lambda > 0 \). Relationships (26) and (27), which make the calculations that follow straightforward, are to be verified once the solution is obtained. Thus, the normalized price signals, informationally equivalent with prices, are:

\[
\tilde{Q}_0 = \tilde{U} - \frac{1}{\gamma S \Omega_0} \tilde{X}_0
\]  

(28)

\[
\tilde{Q}_1 = \tilde{U} - \frac{1}{\gamma S \Omega_1} \tilde{X}_1
\]  

(29)
A.2.1 Period 1

Consider an investor $i$ who, at date $t = 1$, collects $\omega^i_1 \geq 1$ additional signals. At date $t = 1$, investor $i$ chooses $\tilde{D}^i_0$ to maximize expected utility of final wealth:

$$\max_{\tilde{D}^i_0} E \left[ -e^{-\frac{1}{2} \hat{W}^i_2} \left| \tilde{F}^i_1 \right| \right]$$

where the final wealth at date $t = 2$ (at liquidation) is

$$\hat{W}^i_2 = X^i \tilde{P}_0 + \tilde{D}^i_0 (\tilde{P}_1 - \tilde{P}_0) + \tilde{D}^i_1 (\tilde{U} - \tilde{P}_1)$$

and $\tilde{F}^i_1$ represents the total information available at date $t = 1$. This information is given by $\tilde{Z}^i_1, \tilde{Z}^i_0$ (private signals) and $\tilde{Q}^i_1, \tilde{Q}^i_0$ (public signals, informationally equivalent with prices and defined in (28) and (29)). Note that $Z^i_0$ represent only one signal of precision $S$, but $\tilde{Z}^i_1$ represent the average of the $\omega^i_1$ additional signals collected by the investor at date $t = 1$ ($\omega^i_1$ signals of equal precision $S$ are informationally equivalent with a signal equal to their average and having precision $\omega^i_1 S$).

With this information at hand at date $t = 1$, investor $i$ will try to forecast $\tilde{U}$. The state variables corresponding to investor $i$ are therefore

<table>
<thead>
<tr>
<th>Precision → $\tilde{\pi}_i$</th>
<th>$\omega^i_1 S$</th>
<th>$\frac{1}{\omega^i_1}$</th>
<th>$\tilde{\pi}$</th>
<th>$\tilde{\phi}$</th>
<th>$\tilde{\phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable $\downarrow$</td>
<td>$U$</td>
<td>$\tilde{Z}^i_1$</td>
<td>$\tilde{Z}^i_0$</td>
<td>$X_1$</td>
<td>$X_0$</td>
</tr>
<tr>
<td>$U$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z^i_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z^i_0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Q^i_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{\omega^i_1 S}$</td>
<td>0</td>
</tr>
<tr>
<td>$Q^i_0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{\omega^i_1 S}$</td>
<td>0</td>
</tr>
</tbody>
</table>

It is straightforward to calculate

$$K^i_1 = \text{Var}^{-1} \left[ \tilde{Z}^i_1, \tilde{Z}^i_0, \tilde{Q}^i_1, \tilde{Q}^i_0 \right]$$

$$\tilde{\mu}^i_1 = E \left[ \tilde{U} \left| \tilde{Z}^i_1, \tilde{Z}^i_0, \tilde{Q}^i_1, \tilde{Q}^i_0 \right. \right]$$

by using the projection theorem:

**Theorem 4 (Projection Theorem).** Consider a $n$-dimensional normal random variable $(\theta, s) \sim N \left( \begin{bmatrix} \mu_\theta \\ \mu_s \end{bmatrix}, \begin{bmatrix} \Sigma_{\theta,\theta} & \Sigma_{\theta,s} \\ \Sigma_{s,\theta} & \Sigma_{s,s} \end{bmatrix} \right)$

The conditional density of $\theta$ given $s$ is normal with conditional mean

$$\mu_\theta + \Sigma_{\theta,s} \Sigma_{s,s}^{-1} (s - \mu_s)$$

and variance-covariance matrix

$$\Sigma_{\theta,\theta} - \Sigma_{\theta,s} \Sigma_{s,s}^{-1} \Sigma_{s,\theta}$$

provided $\Sigma_{s,s}$ is non-singular.
We get
\[ K_i^1 = H + S \left( 1 + \omega_i^1 \right) + \gamma^2 S^2 \Phi \left( \Omega_0^2 + \Omega_1^2 \right) \]
\[ \mu_i^1 = \frac{1}{K_1} \left[ S \tilde{Z}_0^i + S \omega_i^1 \tilde{Z}_1^i + \gamma^2 S^2 \Phi \left( \Omega_0^2 \tilde{Q}_0 + \Omega_1^2 \tilde{Q}_1 \right) \right] \] (31)

The optimal demand of trader \( i \) in period 1 has a standard form (from the normality of distribution assumption in conjunction with the exponential utility function):
\[ \tilde{D}_i^1 = \gamma K_i^1 \left( \tilde{\mu}_i^1 - \tilde{P}_1 \right) \] (32)

Replace (31) in (32) to obtain
\[ \tilde{D}_i^1 = \gamma \left[ S \tilde{Z}_0^i + S \omega_i^1 \tilde{Z}_1^i + \gamma^2 S^2 \Phi \left( \Omega_0^2 \tilde{Q}_0 + \Omega_1^2 \tilde{Q}_1 \right) - K_i^1 \tilde{P}_1 \right] \] (33)

We can now integrate the optimal demands to get the total demand. We follow the convention used by Admati (1985) that implies \( \int_0^1 \tilde{Z}_j^i = \tilde{U} \), a.s. More important, we have now to keep track of the heterogeneity in information endowments when aggregating all individual demands. In particular, at time \( t = 1 \) there is an infinity of types of investors with respect to their number of signals, and in each such type there is a continuum of investors. Consequently, the total demand at time \( t = 1 \) is
\[ \tilde{D}_1 = \int_0^1 \tilde{D}_i^1 = \sum_{\omega_i^1=1}^{\infty} \pi_1(\omega_i^1) \int_{\omega_i^1} \tilde{D}_i^1 \]
which yields
\[ \tilde{D}_1 = \gamma \left[ S (\Omega_0 + \Omega_1) \tilde{U} + \gamma^2 S^2 \Phi \left( \Omega_0^2 \tilde{Q}_0 + \Omega_1^2 \tilde{Q}_1 \right) - K_1 \tilde{P}_1 \right] \] (34)

where \( K_1 \) is the average precision across the entire population of agents:
\[ K_1 = \sum_{\omega_i^1=1}^{\infty} K_i^1(\omega_i^1) \pi_1(\omega_i^1) = H + S (\Omega_0 + \Omega_1) + \gamma^2 S^2 \Phi \left( \Omega_0^2 + \Omega_1^2 \right) \]

Replace (29) in (34) to obtain
\[ \tilde{D}_1 = \gamma \left[ \left( S \Omega_0 + S \Omega_1 + \gamma^2 S^2 \Phi \Omega_1 \right) \tilde{U} + \gamma^2 S^2 \Phi \Omega_0^2 \tilde{Q}_0 - \gamma \Phi S \Omega_1 \tilde{X}_1 - K_1 \tilde{P}_1 \right] \]

The market clearing condition is \( \tilde{D}_1 = \tilde{X}_0 + \tilde{X}_1 \). Once we impose market clearing, we can use the conjectured \( \tilde{P}_1 \) equation (25) to get the undetermined coefficients \( \varphi_1, \xi_1, \) and \( \alpha_{1,1} \):
\[ \varphi_1 = \frac{S \Omega_1 (1 + \gamma^2 S \Phi \Omega_1)}{K_1}, \]
\[ \xi_1 = \frac{S \Omega_0 (1 + \gamma^2 S \Phi \Omega_0)}{K_1} \]
\[ \alpha_{1,1} = \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1} \]
From these solutions, we can verify that, indeed, \( \frac{\alpha_{1,1}}{\phi_1} = \frac{1}{\gamma \Omega} \). Hence, (27) is now verified. The undetermined coefficients of \( \tilde{P}_1 \) from the conjectured form (23) are

\[
\begin{align*}
\beta_1 &= \frac{K_1 - H}{K_1} \\
\alpha_{1,0} &= \frac{1 + \gamma^2 S \Phi \Omega_0}{\gamma K_1} \\
\alpha_{1,1} &= \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1}
\end{align*}
\]

and thus

\[
\tilde{P}_1 = \frac{K_1 - H}{K_1} \tilde{U} - \frac{1 + \gamma^2 S \Phi \Omega_0}{\gamma K_1} \tilde{X}_0 - \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1} \tilde{X}_1
\]

(35)

which can also be written as

\[
\tilde{P}_1 = \frac{S \Omega_0 + \gamma^2 S^2 \Phi \Omega_0^2}{\gamma K_1} \tilde{Q}_0 + \frac{S \Omega_1 + \gamma^2 S^2 \Phi \Omega_1^2}{\gamma K_1} \tilde{Q}_1
\]

(36)

Furthermore, replacing \( \tilde{P}_1 \) written under the form (36) in the optimal demand written under the form (33) gives the following result:

\[
\tilde{D}_i = \gamma \left[ S \Omega_0 \left( \tilde{Z}_0^i - \tilde{Q}_0 \right) + S \omega_0^i \tilde{Z}_1^i - S \Omega_1 \tilde{Q}_1 - \left( K_1^i - K_1 \right) \tilde{P}_1 \right]
\]

(37)

The coefficient of \( \tilde{P}_1 \) in (37) represents the cumulative informational (dis)advantage of investor \( i \), which we denote thereafter by \( A_1^i \equiv K_1^i - K_1 \). Thus, the optimal demand of investor \( i \) at time \( t = 1 \) is

\[
\tilde{D}_i^i = \gamma \left( S \omega_0^i \tilde{Z}_0^i - S \Omega_0 \tilde{Q}_0 + S \omega_1^i \tilde{Z}_1^i - S \Omega_1 \tilde{Q}_1 - A_1^i \tilde{P}_1 \right)
\]

(38)

where \( \omega_0^i = \Omega_0 = 1 \), but we have included them here to highlight the recursive form of the optimal demand.

**A.2.2 Period 0**

The problem of investor \( i \) at time \( t = 0 \) is

\[
\max_{\tilde{D}_0^i} \mathbb{E} \left[ -e^{-\frac{1}{2} \tilde{W}_2^i} \mid \tilde{Z}_0^i, \tilde{Q}_0 \right]
\]

where the final wealth is given in (30). Observe that, at time \( t = 0 \), investor \( i \) needs to estimate \( \tilde{U} \), \( \tilde{P}_1 \) and \( \tilde{D}_1^i \), after observing \( \tilde{Z}_0^i \) and \( \tilde{Q}_0 \). \( \tilde{P}_1 \) and \( \tilde{D}_1^i \) are given by (35) and (37). The maximization problem then becomes:

\[
\max_{\tilde{D}_0^i} \mathbb{E} \left[ -e^{-\frac{1}{2} \tilde{W}_2^i} \mid \tilde{Z}_0^i, \tilde{Q}_0 \right]
\]

**Lemma 2.** *When an agent builds her portfolio, her future number of signals is irrelevant.*

**Proof.** We prove the claim in a slightly more general context: for the proof only, suppose agent \( i \)
starts with an arbitrary number \( n_0 \) of signals at time \( t = 0 \). The value function \( V^i \) of agent \( i \) is then given by

\[
V^i(n_0, W_0) = e^{-\frac{1}{2}W_0} \max_{\tilde{D}_0} \mathbb{E} \left[ -e^{-\frac{1}{2}(\tilde{D}_0(n_0)(\bar{F}_1-\bar{F}_0)+\tilde{D}_1(n_1)(\bar{U}_1-\bar{F}_1))} | \tilde{Q}_0 \right] 
\]

\[
= e^{-\frac{1}{2}W_0} \max_{\tilde{D}_0(n_0)} \sum_{k=1}^{\infty} \mu_1(k) \mathbb{E} \left[ -e^{-\frac{1}{2}(\tilde{D}_0(n_0)(\bar{F}_1-\bar{F}_0)+\tilde{D}_1(k)(\bar{U}_1-\bar{F}_1))} | \tilde{Q}_0, \tilde{Q}_0; n_1 = k \right]. 
\]

The function \( g \) represents an expectation of an exponential affine quadratic normal variable. To derive its explicit form, we use the following theorem.

**Theorem 5.** Consider a random vector \( z \sim N(0, \Sigma) \). Then,

\[
E \left[ e^{z'Fz+G'z+H} \right] = |I - 2\Sigma F|^{-\frac{1}{2}} e^{\frac{1}{2}G'(I-2\Sigma F)^{-1}G+H}. 
\]

Tedious computations then show that

\[
g \left( n_0, k, \tilde{D}_0 \right) = -|I - 2\Sigma(n_0, k)F(k)|^{-\frac{1}{2}} e^{\frac{1}{2}G(n_0, k, \tilde{D}_0)'(I-2\Sigma(n_0, k)F(k))^{-1}\Sigma(n_0, k)G(n_0, \tilde{D}_0)+H(n_0, k, \tilde{D}_0)} \]

where

\[
\Sigma(n_0, k) = \begin{pmatrix}
\frac{1}{2} K_1(n_0) & \frac{1}{2} K_1^i(k) \\
\frac{1}{2} K_1^i(k) & \frac{1}{2} K_1^i(k)
\end{pmatrix},
\]

\[
F(k) = \begin{pmatrix}
\frac{1}{2} K_1^i(k) & \frac{1}{2} K_1^i(k) \\
\frac{1}{2} K_1^i(k) & \frac{1}{2} K_1^i(k)
\end{pmatrix},
\]

\[
G \left( n_0, k, \tilde{D}_0 \right) = -\frac{\tilde{D}_0}{\gamma} - \frac{K_1^i(n_0)}{K_1^i(n_0) - K_1^i(k)},
\]

\[
H \left( n_0, k, \tilde{D}_0 \right) = -\frac{\tilde{D}_0}{\gamma} - \frac{K_1^i(n_0)}{K_1^i(n_0) - K_1^i(k)} \left( \frac{K_1^i(k)\gamma S((K_0 n_0 + H (\Omega_0 - n_0))\tilde{Q}_0 - K_0 n_0 \tilde{Z}_0)}{1 + \gamma^2 S(\Phi \Omega_0 + H(\Omega_0 - n_0) + n_0 S \Omega_1 (1 + \gamma^2 S(\Phi \Omega_1)) \tilde{Z}_0} \right).
\]

Further computations show that

\[
h(n_0, k) \equiv |I - 2\Sigma(n_0, k)F(k)| = \frac{K_1^i(k)}{K_1^i(n_0) - K_1^i(k)} \left( \frac{H^2\gamma^2\Phi + H(1 + \gamma^2\Phi(K_0 + K_1 - 2H + S\Omega_1))}{n_0(1 + \gamma^2 S(\Phi \Omega_1))^2 + \gamma^2 S(\Phi \Omega_0)(2 + \gamma^2\Phi(K_1 - H + S(\Omega_0 + \Omega_1)))} \right),
\]

and

\[
q \left( n_0, \tilde{D}_0 \right) \equiv \frac{1}{2} G \left( n_0, k, \tilde{D}_0 \right)'(I - 2\Sigma(n_0, k)F(k))^{-1}\Sigma(n_0, k)G \left( n_0, \tilde{D}_0 \right) + H \left( n_0, k, \tilde{D}_0 \right)
\]

44
We integrate the resulting optimal demand and impose market clearing in order to solve for the
where we have added the last term to show that the demand at time 
for all investors:

\[ V^i(n_0, W_0) = e^{-\frac{1}{\gamma} W_0} \left( \sum_{k=1}^{\infty} \mu_1(k) h(n_0, k) \right) \max_{D_0(n_0)} -e^{q(n_0,D_0)} \]  

(40)

and it follows that her portfolio decision is independent of her expectation regarding her future number of signals. ■

To obtain agent \( i \)'s optimal demand, we solve the problem in (40) and impose the first-order condition

\[ \frac{\partial}{\partial D_0} q \left( n_0, D_0^i \right) = 0. \]

We integrate the resulting optimal demand and impose market clearing in order to solve for the undetermined coefficients of \( P_0 \), i.e., \( \beta_0 \) and \( \alpha_{0,0} \). The solutions for these coefficients are:

\[ \beta_0 = \frac{K_0 - H}{K_0} \]

\[ \alpha_{0,0} = \frac{1 + \gamma^2 S \Phi \Omega_0}{\gamma K_0} \]  

(41)

where

\[ K_0 = K_0^i = H + S + \gamma^2 S^2 \Phi \Omega_0^2 \]

\[ \bar{\mu}_0^i = \frac{1}{K_0^i} \left( S \bar{Z}_0^i + \gamma^2 S^2 \Phi \Omega_0^2 \bar{Q}_0 \right) \]  

(42)

We have \( K_0 = K_0^i \) in (42) because investors start with homogeneous information endowments at time 0, i.e., all have one signal. In other words, the cumulative informational (dis)advantage is zero for all investors: \( A_t^i = 0 \), \( \forall i \). The optimal demand of investor \( i \) at time \( t = 0 \) is

\[ D_0^i = \gamma \left( S \bar{w}_0^i \bar{Z}_0^i - S \Omega_0 \bar{Q}_0 - A_0^i \bar{P}_0 \right) \]  

(43)

where we have added the last term to show that the demand at time \( t = 0 \) takes a similar form with the demand at time \( t = 1 \), expressed in equation (38). At this point, we can use (41) and (42) to verify that, indeed, \( \frac{\alpha_{0,0}}{\beta_0} = \frac{1}{\gamma S \Omega_0} \). Hence, (26) is now verified. The solution can then be written in a
recursive form and extended to more than 2 trading periods, as done in Theorem 1.

A.2.3 Optimal Trading Strategy

The optimal trading strategy of investor \( i \) at time \( t = 1 \) follows from (38) and (43):

\[
\Delta \tilde{D}_i^1 \equiv \tilde{D}_i^1 - \tilde{D}_0^i = \gamma \left( S \omega_i^1 \tilde{Z}_i^1 - S \Omega_1 \tilde{Q}_1 - A_i^1 \tilde{P}_1 + A_0^i \tilde{P}_0 \right) \\
= \gamma \left[ S \omega_i^1 \tilde{Z}_i^1 - S \Omega_1 \tilde{Q}_1 - A_i^1 \left( \tilde{P}_1 - \tilde{P}_0 \right) - \left( A_i^1 - A_0^i \right) \tilde{P}_0 \right]
\]

The cumulative informational (dis)advantage at time \( t = 1 \), \( A_i^1 \), has a recursive form:

\[
A_i^1 = A_i^0 + S \omega_i^1 - S \Omega_1 = A_i^0 + a_i^1
\]

where \( a_i^1 \equiv S \omega_i^1 - S \Omega_1 \) is the marginal informational (dis)advantage of trader \( i \) at time \( t = 1 \).

Thus, the optimal trading strategy becomes:

\[
\Delta \tilde{D}_i^1 = \gamma \left[ \left( S \omega_i^1 \tilde{Z}_i^1 - S \Omega_1 \right) \left( \tilde{Q}_1 - \tilde{P}_0 \right) + S \Omega_1 \left( \tilde{Z}_i^1 - \tilde{P}_0 \right) - S \Omega_1 \left( \tilde{Q}_1 - \tilde{P}_0 \right) - A_i^1 \left( \tilde{P}_1 - \tilde{P}_0 \right) \right] \quad (44)
\]

Furthermore, it can be verified that

\[
\tilde{Q}_0 = \frac{K_0 \tilde{P}_0}{S \Omega_0 + \gamma^2 S^2 \Phi \Omega_0^2} \\
\tilde{Q}_1 = \frac{K_1 \tilde{P}_1 - K_0 \tilde{P}_0}{S \Omega_1 + \gamma^2 S^2 \Phi \Omega_1^2}
\]

Replacing \( \tilde{Q}_1 \) in (44) yields

\[
\Delta \tilde{D}_i^1 = \gamma \left[ a_i^1 \left( \tilde{Z}_i^1 - \tilde{P}_0 \right) + S \Omega_1 \left( \tilde{Z}_i^1 - \tilde{P}_0 \right) - \frac{K_1}{1 + \gamma^2 S \Phi \Omega_1} \left( \tilde{P}_1 - \tilde{P}_0 \right) - A_i^1 \left( \tilde{P}_1 - \tilde{P}_0 \right) \right]
\]

This is the same recursive form as in (9) and thus the proof of Theorem 1 is complete.  \( \square \)
A.3 Formula for $\rho_t$ and Proof of Proposition 3

$$\rho_t = \frac{\left(\frac{1}{K_t} - \frac{1}{K_{t+1}}\right) \left(\frac{1}{K_{t-1}} - \frac{1}{K_t}\right) \left(H + \sum_{j=0}^{t-1}(1 + \gamma^2 S \Omega_j \Phi)^2\right) - \frac{1}{\gamma^2 K_t \Phi} \left(1 + \gamma^2 S \Omega_t \Phi\right)^2}{\left(\frac{1}{K_{t-1}} - \frac{1}{K_t}\right)^2 \left(H + \frac{1}{\gamma^2 \Phi} \sum_{j=0}^{t-1}(1 + \gamma^2 S \Omega_j \Phi)^2\right) + \frac{1}{\gamma^2 K_t \Phi} \left(1 + \gamma^2 S \Omega_t \Phi\right)^2}.$$ \hspace{1cm} (45)

Proof of Proposition 3

Shutting down social interactions, prices are obtained as a special case of (2) when the average incremental number of signal satisfies $\Omega_t \equiv 1$; accordingly, the numerator in (45) simplifies to

$$-\frac{HS \left(\gamma^2 S \Phi + 1\right)^2}{\gamma^2 \Phi \left(H + St \left(\gamma^2 S \Phi + 1\right)\right) \left(H + S(t + 1) \left(\gamma^2 S \Phi + 1\right)\right)^2 \left(H + S(t + 2) \left(\gamma^2 S \Phi + 1\right)\right)} < 0.$$ 

This expression is negative for any time $t$. 
A.4 Appendix for Section 4.2.2

This Appendix mainly follows Andrei (2013). Consider the following processes for dividends and noisy supply:

\[ D_t = \kappa_d D_{t-1} + \varepsilon_d^t \]  
\[ X_t = \kappa_x X_{t-1} + \varepsilon_x^t \]  

(46)  
(47)

where \( 0 \leq \kappa_d \leq 1 \) and \( 0 \leq \kappa_x \leq 1 \). The dividend and supply innovations are \( i.i.d. \) with normal distributions: \( \varepsilon_d^t \sim N(0, 1/H) \) and \( \varepsilon_x^t \sim N(0, 1/\Phi) \). There is one riskless bond assumed to have an infinitely elastic supply at positive constant gross interest rate \( R \).

The economy is populated by a continuum of rational agents, indexed by \( i \), with CARA utilities and common risk aversion \( 1/\gamma \). Each agent lives for two periods, while the economy goes on forever (overlapping generations). All investors observe the past and current realizations of dividends and of the stock prices. Additionally, each investor observe an information signal about the dividend innovation 3-steps ahead:

\[ \tilde{z}_t = \varepsilon_d^{t+3} + \varepsilon_x^t \]

As time goes by, investors share their private information at random meetings. The information structure and the probability density function over the number of private signals is described in Andrei (2013). As usual in noisy rational expectations, we conjecture a linear function of model innovations for the equilibrium price:

\[ P_t = \alpha D_t + \beta X_{t-3} + (a_3 \ a_2 \ a_1)\varepsilon_d^t + (b_3 \ b_2 \ b_1)\varepsilon_x^t \]

Proposition 1 in Andrei (2013) describes the rational expectations equilibrium, which is found by solving a fixed point problem provided by the market clearing condition. Infinite horizon models with overlapping generations have multiple equilibria (there are \( 2^N \) equilibria for a model with \( N \) assets). The model studied here has 2 equilibria, one low volatility equilibrium and one high volatility equilibrium. We focus on the low volatility equilibrium, which is the limit of the unique equilibrium in the finite version of the model.

To understand how the two equilibria arise, let’s assume that there is no private information. In this case, the equilibrium price has a closed form solution:

\[ P_t = \frac{\kappa_d}{R - \kappa_d} D_t - \sum \frac{\kappa_x^3}{R - \kappa_x} X_{t-3} - \sum \frac{\kappa_x^2}{R - \kappa_x} \varepsilon_x^{t-2} - \sum \frac{\kappa_x}{R - \kappa_x} \varepsilon_x^{t-1} - \sum \frac{1}{\gamma} \frac{\kappa_x}{R - \kappa_x} \varepsilon_x^t \]

where \( \Sigma \equiv (\alpha + 1)^2 \sigma_d^2 + b_1^2 \sigma_x^2 \). Thus, the coefficient \( b_1 \) has to solve a quadratic equation:

\[ b_1 = -\frac{\gamma}{R - \kappa_x} \left[ \left( \frac{R}{R - \kappa_d} \right)^2 \sigma_d^2 + b_1^2 \sigma_x^2 \right] \]

For different parameter values, the above quadratic equation can have two solutions, one solution, or none. In this particular example (no private information), the autocovariance of stock returns,
Cov \((P_{t+1} - P_t, P_{t+2} - P_{t+1})\), is

\[
\text{Cov} \left( P_{t+1} - P_t, P_{t+2} - P_{t+1} \right) = -\alpha^2 \sigma^2_d \frac{1 - \kappa_d}{1 + \kappa_d} + \beta^2 (\kappa_x - 1)^2 \sigma^2_x \frac{1}{1 - \kappa_x^2} +
\]

\[
\left( \begin{array}{cccc}
\beta - b_3 & b_3 - b_2 & b_2 - b_1 & b_1
\end{array} \right) \left( \begin{array}{c}
-\beta(1 - \kappa_x) \\
\beta - b_3 \\
b_3 - b_2 \\
b_2 - b_1
\end{array} \right)
\]

It can be shown numerically that this covariance is generally negative when \(\kappa_d < 1\) and \(\kappa_x < 1\). In the random walk specification (46) - (47), the covariance is zero.

Now, if agents receive private information, the model has to be solved numerically using the methodology described in Andrei (2013). More precisely, \(\alpha, \beta, a,\) and \(b\) solve the following equations:

\[
(\alpha + 1)\kappa_d - R\alpha = 0
\]

\[
\bar{K}_t \beta \kappa_x - \bar{K}_t R \beta - \frac{1}{\gamma} \kappa_x^3 = 0
\]

\[
\bar{K}_t b^* \mathbb{B}^{-1} A + \bar{L}_t \mathbb{H} - \bar{K}_t R a = 0_{1 \times 3}
\]

\[
\bar{K}_t b^* + \bar{L}_t \mathbb{B}^* - \bar{K}_t R b - \frac{1}{\gamma} \left( \kappa_x^2 \kappa_x \right) = 0_{1 \times 3}
\]

where \(\bar{K}_t, b^*, \mathbb{B}, A, \bar{L}_t, \mathbb{H},\) and \(\mathbb{B}^*\) are defined in Appendix A.3 of Andrei (2013).
A.5 Appendix for Section 5

To prove Theorem 3, we adapt the expression for the price $\tilde{P}$ in Brennan and Cao (1997) and write

$$\tilde{P}_t = \beta_t \tilde{U} + \alpha_t \tilde{V} + \sum_{j=0}^{t-1} \xi_{j,t} \tilde{Q}_j - \gamma_t \tilde{X}_t.$$

The price is informationally equivalent to

$$\tilde{Q}_t = \frac{1}{\beta_t} \tilde{P}_t - \sum_{j=0}^{t-1} \xi_{j,t} \tilde{Q}_j = \tilde{U} + \frac{\alpha_t}{\beta_t} \tilde{V} - \gamma_t \tilde{X}_t.$$

Furthermore, we can write agent $i$’s individual demand as

$$\tilde{D}_t^i = \omega_t^i \tilde{P}_t + \sum_{k=0}^{t} \lambda_t^i \tilde{Q}_k + \sum_{k=0}^{t} \theta_t^i \tilde{Z}_k.$$

By the law of large numbers, we have that $\int_{i \in [0,1]} \tilde{Z}_k^i d\mu(i) = \tilde{U} + \tilde{V}$. As a result, when we aggregate individual demands, we obtain

$$\int_{i \in [0,1]} \tilde{D}_t^i d\mu(i) = \tilde{\omega}_t \tilde{P}_t + \sum_{k=0}^{t} \lambda_t^k \tilde{Q}_k + \sum_{k=0}^{t} \theta_t^k \tilde{U} + \sum_{k=0}^{t} \theta_t \tilde{V}$$

where $\tilde{\omega}_t = \sum_{i} \pi_t(k) \omega_t^i(k)$, $\tilde{\lambda}_t = \sum_{i} \pi_t(k) \lambda_t^i(k)$, and $\tilde{\theta}_t = \sum_{i} \pi_t(k) \theta_t^i(k)$.

Imposing market clearing, we have

$$\sum_{k=0}^{t} \tilde{X}_k - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{U} - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{V} = \tilde{\omega}_t \tilde{P}_t + \sum_{k=0}^{t} \tilde{\lambda}_k \tilde{Q}_k.$$

Substituting

$$\tilde{P}_t = \beta_t \tilde{Q}_t + \sum_{j=0}^{t-1} \xi_{j,t} \tilde{Q}_j$$

into the above equation, we obtain

$$\sum_{k=0}^{t} \tilde{X}_k - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{U} - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{V} = \tilde{\omega}_t (\beta_t \tilde{Q}_t + \sum_{j=0}^{t-1} \xi_{j,t} \tilde{Q}_j) + \sum_{k=0}^{t} \tilde{\lambda}_k \tilde{Q}_k.$$

Furthermore, notice that

$$\tilde{X}_k = \frac{\beta_k}{\gamma_k} \left( \tilde{U} + \frac{\alpha_k}{\beta_k} \tilde{V} - \tilde{Q}_k \right).$$

Substituting and regrouping, we obtain

$$\tilde{X}_t + \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \left( \tilde{U} + \frac{\alpha_k}{\beta_k} \tilde{V} - \tilde{Q}_k \right) - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{U} - \sum_{k=0}^{t} \tilde{\theta}_k \tilde{V} = (\tilde{\omega}_t \beta_t + \tilde{\lambda}_t) \tilde{Q}_t + \sum_{j=0}^{t-1} (\xi_{j,t} \tilde{\omega}_t + \tilde{\lambda}_j \tilde{Q}_j),$$

50
or, equivalently,

\[ \ddot{X}_t + \left( \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k \right) \ddot{U} + \left( \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t-1} \bar{\theta}_k \right) \ddot{V} - \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \ddot{Q}_k \equiv f(\{\tilde{Q}_j\}_{j=0}^t). \]

The right-hand side of this equation is only a function of \(\{\tilde{Q}_j\}_{j=0}^t\). By separation of variables, the left-hand side must also be a function of \(\{\tilde{Q}_j\}_{j=0}^t\) only. Hence, it must be that

\[- \left( \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k \right) \left( \ddot{U} + \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t-1} \bar{\theta}_k \right) \ddot{V} - \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \ddot{X}_t \right) \equiv \ddot{Q}_t. \]

This equality holds if and only if

\[ \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t-1} \bar{\theta}_k = \frac{\alpha_t}{\beta_t} \]

and

\[ \frac{1}{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k} = \frac{\gamma_t}{\beta_t}. \]

Without loss of generality, we set

\[ \frac{\gamma_t}{\beta_t} = \frac{1}{rS\Omega_t} \]

so that

\[ \sum_{k=0}^{t} \bar{\theta}_k = \sum_{k=0}^{t} \frac{\beta_k}{\alpha_k} = rS \sum_{k=0}^{t} \Omega_k. \]

and

\[ \frac{\alpha_t}{\beta_t} = \frac{\eta \Lambda_t}{rS\Omega_t} \]

so that:

\[ \eta \sum_{k=0}^{t} \bar{\theta}_k = \sum_{k=0}^{t} \frac{\beta_k}{\alpha_k} \bar{\theta}_k = \sum_{k=0}^{t} \Lambda_k. \]

The system of equations in (15) follows. This system of equations is a fixed point: to solve it, we solve the problem recursively (as in Appendix A.2, except accounting for the rumor) over 4 periods. We then start with guess values for \(\{\Omega_j\}_{j=0}^3\) and \(\{\Lambda_j\}_{j=0}^3\) and get, through the fixed point in (15), new values for these coefficients. Iterating and invoking the Contraction-Mapping Theorem, we obtain the equilibrium coefficients. ■