Credit Default Swap Spreads and Systemic Financial Risk

Stefano Giglio*
University of Chicago, Booth School of Business
November 2011

Abstract

This paper measures joint default risk of financial institutions exploiting information about counterparty risk in credit default swaps (CDS). A CDS contract written by a bank to insure against the default of another bank is exposed to the risk that both default together. From CDS spreads we can then directly learn about the default risk of pairs of banks. Since information about pairwise default risk does not fully characterize multiple default risk, I derive the tightest bounds on the probability that many banks fail simultaneously.

1 Introduction

During periods of financial distress, the probability that several financial institutions default together – systemic default risk – can increase dramatically and abruptly since financial intermediaries are highly interconnected and are exposed to common shocks. It is then important to construct measures of systemic risk that capture the joint default risk even at short horizons, and can react quickly when new information arrives.

Market-based measures, the most widely used, estimate the joint distribution of defaults from the prices of financial instruments like credit default swaps (CDS)\(^1\). Relative to reduced-form measures, which estimate the joint default probabilities using historical data on returns,\(^2\) they have the advantage of immediately incorporating new information. In addition, they do not need to rely on a few data points to estimate the tails of the distributions, as joint defaults are extremely rare events. Relative to structural measures of default based on the Merton (1974) model,\(^3\) they require less stringent assumptions about the liability structure of the financial institutions.

Figure 1 plots two simple examples of market-based measures: the average 5-year bond yield spread and the average 5-year CDS spread of the 15 financial institutions that are the largest issuers of CDS protection. These measures reflect the average default risk of these institutions, and behave similarly to other widely used market-based measures, like the ones reported in the IMF Global Financial Stability Report and in the ECB Financial Stability Review. All these measures suggest an increase in systemic risk starting in August 2007, followed by several episodes in which systemic risk spiked (such as around March 2008, September 2008 and then March 2009), and a final drop after April 2009.

Measures like these can be misleading for two reasons. First, they ignore the effects of counterparty risk on the prices of these securities. Second, the securities employed reflect individual default risk rather than joint default risk. To extract a measure of joint default risk one must then impose strong modeling assumptions

---

\(^1\)A credit default swap is an insurance contract against the default of a firm, for example a financial institution. The CDS spread corresponds to the yearly insurance premium. See section 2 for details on the contract.

\(^2\)Acharya et al. (2010) and Adrian and Brunnermeier (2009) are recent examples of measures based on historical data.

\(^3\)For example, Lehar (2005) and Gray et al. (2008) apply the Merton (1974) model to estimating joint default probabilities.
about the joint distribution function of defaults.

This paper makes three contributions to the measurement of systemic risk using
market prices and applies them by measuring systemic default risk during the recent

First, it shows that the presence of *counterparty risk* in over-the-counter mar-

Second, this paper shows that we can in fact exploit the presence of counterparty

risk in CDSs to learn about the joint default events of pairs of banks. Since the prices

of bonds issued by banks are not affected by counterparty risk, we can use them to

learn directly about the banks’ individual default probabilities. Joint default risk of

the bond issuer with another bank can then be inferred from the discount at which

CDS insurance against it sold by the other bank sells relative to the bond price. This

enriches significantly the information set we can employ to construct measures of

systemic risk.

Figure 1: Average 5-year bond yield and CDS spread for the 15 financial intermediaries
most active in the CDS market, and rescaled difference.
Third, I present a new methodology to construct these measures. Most market-based measures extract individual default probabilities from CDS spreads or bond prices. To fully characterize the joint distribution function, they make strong assumptions about the shape of this distribution, for example imposing that it is multivariate normal or student-t. All the information about tail events these measures employ comes only from the individual default probabilities, and the results depend heavily on the assumptions on the joint distribution function. I show that it is instead possible to learn about systemic risk making in fact no assumptions about the shape of the joint distribution function. In particular, I demonstrate how to construct the tightest possible bounds on systemic default risk of degree $r$ – defined as the probability that at least $r$ banks default together – consistent with the available information set containing marginal and pairwise default probabilities. While we cannot completely pin down systemic risk, we can nonetheless restrict the range of joint default probabilities consistent with the observed prices. In addition, since we can solve for the configuration of the network that attains the upper and lower bounds, we can learn the contribution to systemic risk of each bank in the best and worst systemic risk scenarios.

I then apply this methodology and compute the tightest bounds on the probability that at least $r$ out of 15 large global financial institutions default within a month, for different $r$’s, using CDS and bond data from 2004 to 2010. The calculations show that the information contained in these bounds allows us to characterize in interesting ways the evolution of systemic risk during the financial crisis. In particular, contrary to the other market-based measures, which indicate a sharp increase in systemic risk already in 2007, we can exclude a large increase in systemic default risk before Bear Stearns’ failure in March 2008. In several episodes observed spikes in CDS spreads and bond yields can be attributed more to spikes in idiosyncratic risk than in systemic risk. This is the case in the month before Bear’s collapse and, partly, in the month after Lehman’s default.

Three limitations affect the empirical construction of the bounds. First, the presence of an unobserved liquidity process in the bond market confounds the estimation of individual default probabilities. Second, for every bank, I observe only an average

---

4If some additional parameters need to be specified after choosing a certain copula (for example correlations), these are usually estimated from historical data. Huang, Zhou and Zhu (2009), Avesani, Pascual and Li (2006) and Segoviano and Goodhart (2009) are prominent examples of this approach.
of the CDS quotes across counterparties. Third, I obtain bounds on risk neutral, not objective, probabilities of systemic events.\textsuperscript{5} Risk-neutral probabilities are interesting since they reveal the markets' combined perception of the probability and severity of these states of the world. In addition, they can be considered upper bounds on the objective default probabilities, as long as default states are states with high marginal utility. Notwithstanding these limitations, the bounds allow us to learn significant information about the evolution of systemic risk during the financial crisis.

The paper proceeds as follows. Section 2 discusses the role of counterparty risk in credit default swap contracts and shows how it can be used to learn about joint default probabilities of pairs of banks. Section 3 presents the theory of the optimal probability bounds and discusses their properties. Section 4 computes the bounds on systemic risk between 2004 and 2010. Section 5 concludes.

\section{Credit Default Swaps, Counterparty Risk and Measures of Systemic Risk}

In this section I discuss the role of counterparty risk in CDS markets both as an element of bias in measures of systemic risk and as a valuable additional source of information about joint default risk of pairs of financial institutions.

\subsection{The Credit Default Swaps Market – an Introduction}

In a typical CDS contract, the protection seller offers the protection buyer insurance against the default of an underlying bond issued by a certain company, the reference entity. The seller commits to buy the bond from the protection buyer for a price equal to its face value in the event of default by the reference entity or other defined credit event. In some cases, the contract is cash settled, so that the seller directly pays the buyer the difference between the face value and the recovery value of the bond (see Appendix C for more details on the contract). In exchange for the insurance, the buyer pays a quarterly premium, the \textit{CDS spread}, quoted as an annualized percentage of the notional value insured. If default occurs, the contract terminates, and the quarterly

\textsuperscript{5}Anderson (2009) underlines the differences between the two by comparing risk-neutral default processes obtained from CDS spreads with objective processes obtained using historical data on defaults.
payments are interrupted. If default does not occur during the life of the contract, the contract terminates at its maturity date.

While in general these contracts are traded over the counter and can be customized by the buyer and the seller, in the recent years they have become more standardized, following the guidelines of the International Swaps and Derivatives Association (ISDA). The CDS market is quite liquid, with low transaction costs to initiate a contract with a market maker on short notice, and with numerous dealers posting quotes (see Blanco et al. (2003) and Longstaff et al. (2005)). Reliable quotes for the 5-year maturity CDS can be obtained through several financial data firms (e.g. Bloomberg, Datastream, Markit).

The CDS market has grown quickly in the last few years. Notional exposures grew from about $5 trillion in 2004 to around $60 trillion at its peak in 2007, and despite the financial crisis, the total notional exposure is still around $40 trillion. The high liquidity of the CDS market has made it the easiest way to adjust exposures to credit risk, and has been the primary reason for its growth. As a consequence, rather than trading in the bond market or canceling agreements already in place, adjustments of credit exposures have been achieved by simply entering new CDS contracts, possibly offsetting existing ones.

At the center of this network of CDS contracts, a few main dealers operated with very high gross and low net exposures, emerging as the main counterparties in the market. For example, Fitch Ratings (2006) states that in 2006 the top 10 counterparties accounted for about 89% of the total protection sold. With the crisis, the market concentrated even more after some of its key players disappeared.

### 2.2 Counterparty Risk

Traded over the counter, a CDS contract involves counterparty risk: the protection seller may go bankrupt during the life of the CDS and therefore might not be able to comply with the commitments implied by the contract. When the counterparty goes bankrupt, the contract terminates and a claim arises between the two parties equal to the current value of the contract. A buyer of a CDS contract is then exposed to the

---

6The role of counterparty risk in CDS spreads has been studied by Hull and White (2001), Jarrow and Turnbull (1995), Jarrow and Yu (2001), and more recently, in the context of rare disaster risk, by Barro (2010). To mitigate counterparty risk - which stems mainly from the OTC nature of the contract - there are now several proposals to create a centralized clearinghouse. For a detailed discussion, see Duffie and Zhu (2010).
risk that when the counterparty defaults the credit risk of the reference entity is higher than it was when he started the contract. In this case, the contract is in the money and the buyer has a claim against the bankrupt counterparty, which might be difficult to collect. The larger the increase in the CDS spread of the reference entity (its credit risk) when the seller defaults, the larger the amount the seller owes the buyer. In the extreme case, in which the bankruptcy of the seller occurs simultaneously with the default of the reference entity, the payment due to the buyer would be equal to the full notional value of the CDS, and the buyer would have a very large claim against the bankrupt counterparty. The buyer risks not to get paid exactly in the one state where the CDS contract is supposed to pay off.

Note that the situations in which the CDS buyer is exposed to a large loss following the bankruptcy of the seller and the reference entity are not confined to the case of simultaneous default. For example, the reference entity might default after the seller defaults, but the default risk of the reference already jumps when the seller goes bankrupt. In this case the contract will be highly in the money and the amount the buyer can claim against the bankrupt seller can be very high. Alternatively, the buyer might suffer a large loss if the default of the reference entity triggers the subsequent default of the counterparty, for example if the latter is not adequately hedged. All these cases – collectively referred to as double default cases – are relevant for the value of the CDS to the buyer even though the two defaults do not occur simultaneously: in all these cases the buyer finds herself with a large amount owed by the bankrupt counterparty.

The value of the CDS protection to the buyer crucially depends on how much of this claim the buyer can expect to recover from the counterparty in bankruptcy. Like other derivatives, CDS claims are protected by “safe harbor” provisions of the bankruptcy law. These exempt claimants from the automatic stay of the assets of the firms, so that the buyers can immediately seize any collateral that has been posted for them, as discussed in detail in the next section. Additionally, positions across different derivatives with the same counterparty can be netted against each other. The latter fact potentially increases the recovery in case of counterparty default, but only if the buyer finds herself with large enough out-of-the-money positions with the seller, when the seller defaults, to hedge counterparty risk.

For all amounts that remain after netting and seizing posted collateral, the buyer is an unsecured creditor in the bankruptcy process, and as such is exposed to potentially
large losses.

2.3 Collateral Agreements and Counterparty Risk

In order to protect buyers against counterparty risk, some (but not all) CDS contracts involve a collateral agreement. Under the standard collateral agreement, a small margin is posted at the inception of the contract. Subsequent collateral calls are tied to changes in the value of the CDS contract, as well as to the credit risk of the seller.

While helpful in reducing counterparty exposure, collateral agreements cannot completely eliminate counterparty risk. First, according to the ISDA Margin Survey 2008, only about 66% of the nominal exposure in credit derivatives, of which CDSs are the most important type, had a collateral agreement at all in 2007 and 2008. In addition, as reported in the ISDA Survey, collateral agreements were employed much less frequently when the counterparty was a large dealer. Second, margins were often posted at a less than daily frequency. The 2011 ISDA Margin Survey reports that only 61% of all trades executed by large dealers had daily collateral adjustment, most of which concentrated in inter-dealer trades. For 26% of trades by large dealers margins were not adjusted regularly. Third, often collateral posted was lower than the current value of the position. Even the buyer that most aggressively called for collateral during the crisis, Goldman Sachs, was not covered completely on its CDS exposures with other large dealers (in particular, AIG), let alone the potential exposure in case of sudden default of the reference entities. Finally, the sudden nature of default events imposes counterparty risk even when the full current value of the positions is covered by collateral.

The Lehman bankruptcy is an interesting example of the limits of collateralization. Before the weekend of September during which Lehman collapsed and two other large financial institutions were bailed out (Merrill Lynch and AIG), low CDS spreads suggested a small risk of bankruptcy of those dealers. The joint shock to the three institutions was certainly not anticipated, and the amount of collateral posted at that point was small. As it turned out, thanks to the government bailout a double default

---

7 Appendix A reports additional evidence on the role of collateral.
8 This number went up to 93% in 2011.
9 The documents reported in Appendix A refer specifically to a large amount ($22bn) of CDS protection bought by Goldman from AIG on super-senior tranches of CDOs, but arguably similar practices were used on all credit derivatives instruments.
10 For example, someone who bought a 5-year Lehman CDS a month before its default would have
event did not materialize. However, these events show that the risk of simultaneous collapse of several banks is relevant, and that standard collateralization practices would not have prevented large losses to buyers of CDS contracts, had the government decided not to intervene.

The presence of collateral agreements improves but does not solve the problem of counterparty risk related to double default: the amount owed by the CDS seller in case of double default would be very large, and for the reasons specified above it might easily exceed the collateral posted. Buyers of CDSs were aware of this residual counterparty risk, as shown in several documents reported in Appendix A, and frequently believed that the best way to reduce their remaining counterparty exposure was to buy additional CDS protection against their counterparty, which directly increased the cost of buying CDS protection.

2.4 The pricing of counterparty risk: a simple example

A simple two-period example of the pricing of bonds and CDSs can be useful to understand the role of counterparty risk on CDS prices. Consider a group of $N$ dealers that have each issued a zero-coupon bond with a face value of $1$ maturing at time $1$, and the CDS contract written at time $0$ by each of them against the default of each other dealer. Call $A_i$ the event of default of institution $i$ at time $1$. Call $P(A_i)$ the probability of default of bank $i$, and $P(A_i \cap A_j)$ the probability of joint default of $i$ and $j$ at time $1$. All probabilities are risk-neutral. Call $R$ the expected recovery rate on the unsecured bond in case of default, and suppose that in the event of joint default the CDS claim recovers a fraction $S$. Note that $S \geq R$ since as an unsecured creditor the buyer of CDS protection would obtain $R$ in bankruptcy court, but netting and collateralization mean that she might recover part of the amount even before going to court. Finally, assume that the risk-free rate between periods $0$ and $1$ is zero.

In this setting, the price of the bond issued by $i$, $p_i$, is determined as:

$$p_i = (1 - P(A_i)) + P(A_i) R$$

been in the money, on Friday September 12th, for about 15 cents on the dollar – this would have been the amount of collateral in her possession if the contract was fully collateralized.

11For example, Barclays Capital issued a report in February 2008, “Counterparty Risk in Credit Markets”, precisely on the effect of counterparty risk on CDS prices.
If there is no counterparty risk in the CDS contract, the insurance premium $z_i$, or CDS spread, paid at time 0 to insure that bond is:

$$z_i = P(A_i)(1 - R)$$

A theoretical arbitrage relation then holds between the bond and the CDS (Longstaff et al. (2005)): $z_i = 1 - p_i$.

Consider now the case in which there is counterparty risk in the CDS contract. Then, the spread paid to buy insurance from $j$ against $i$’s default will be:

$$z_{ji} = [P(A_i) - P(A_i \cap A_j)](1 - R) + P(A_i \cap A_j)(1 - R)S$$

$$= [P(A_i) - (1 - S)P(A_i \cap A_j)](1 - R)$$

(2)

since the buyer of protection obtains the full payment $(1 - R)$ if the reference entity defaults alone, and only a fraction $S$ of it otherwise. The spread $z_{ji}$ decreases with the probability of joint default $P(A_i \cap A_j)$; the arbitrage relation with the bond is broken. Note that the order of magnitude of counterparty risk could in theory be as high as the spread itself. If defaults are independent we have $P(A_i \cap A_j) = P(A_i)P(A_j)$, but for financial intermediaries defaults are not likely to be independent even at short horizons. Therefore, the probability of the joint default can be much larger than the product of the marginal probabilities.

Equation (2) shows that without additional information, it is not possible to distinguish the component of the CDS spread that comes from the risk of the reference entity, $P(A_i)$, from the joint default risk with the counterparty, $P(A_i \cap A_j)$. Therefore, unless one makes stringent assumptions, it is not possible to detect counterparty risk using CDS data alone. Gandhi et al. (2011) for example study the cross-section of CDS prices across counterparties under the assumption that given a reference entity $i$, counterparty risk with each counterparty $j$ is captured by $P(A_i)$, not by $P(A_i \cap A_j)$ as the theory would predict. In the empirical analysis of Section 4, in which I allow for marginal and joint default risk to move independently, I find that they often behave in very different ways.

From equation (2) we also see that ignoring counterparty risk biases the estimates of default probability downwards. In particular, measures of systemic risk obtained by averaging CDS spreads of banks (as in Figure 1) will depend negatively on averages
of \( P(A_i \cap A_j) \) across \( i \) and \( j \). If joint default risk across the network increases, these measures would then decrease, and suggesting a decrease in systemic risk.

3 Construction of the probability bounds

This section develops the probability bounds on systemic risk for a network of banks in which bond and CDS prices are observed. I start with an introductory example with three banks, and then show how to construct the bounds in the general case of \( N \) banks using linear programming.

3.1 Introductory example

Consider a two-period setting, and suppose that the financial sector consists of only three intermediaries – banks 1, 2 and 3. Protection against the default of \( i \in I \equiv \{1, 2, 3\} \) must be bought from a bank \( j \in I \setminus i \), i.e. one of the other two intermediaries. As shown by the pricing formulas (1) and (2), if we observe all bond prices \( p_i \) and all CDS spreads \( z_{ji} \), we can learn the marginal default probabilities of each bank as well as the pairwise default probabilities for each pair \((i, j)\) of banks. For example, suppose we observe:

\[
P(A_i) = 0.2 \ \forall i
\]

\[
P(A_1 \cap A_2) = P(A_2 \cap A_3) = 0.07, \ P(A_1 \cap A_3) = 0.01
\]

and I want to measure \( P_r \), the probability of joint default of at least \( r \) financial intermediaries. With only three banks, I obtain the following three potential measures of systemic risk:

\[
P_1 = P(A_1 \cup A_2 \cup A_3)
\]

\[
P_2 = P((A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3))
\]

\[
P_3 = P(A_1 \cap A_2 \cap A_3)
\]

At first sight, one might think that if we observed all bond prices and all CDS
spreads, thus learning $P(A_i)$ and $P(A_i \cap A_j)$ for each $i$ and $j$, we would be able to completely pin down the systemic probabilities $P_1$, $P_2$ and $P_3$. However, information about individual and pairwise probabilities is insufficient to fully characterize $P_r$. In Figure 2, a Venn diagram in which areas represent probabilities, the area of each event is the same across the two panels, so the marginal probabilities of defaults are the same (all 0.2). The same is true for the pairwise default probabilities (0.07, 0.07, 0.01). However, it is easy to see that $P_3$, the intersection of all three events, is positive (0.01) in the left panel and zero in the right panel; similarly, $P_1$ and $P_2$ are different across panels.

![Figure 2: Example of the relation between low-order and high-order probabilities.](image)

Knowledge of marginal and pairwise probabilities, however, allows us to put bounds on other probabilities, and in particular on systemic default risk. For example, because we know $P(A_1 \cap A_2 \cap A_3) \leq P(A_1 \cap A_3)$, we can immediately establish $P_3 \leq 0.01$. Finding the other bounds is more complicated. The object of the rest of this section is to show how to find the tightest possible ones. For this example, they are:

\[
0.45 \leq P_1 \leq 0.46 \\
0.13 \leq P_2 \leq 0.15 \\
0 \leq P_3 \leq 0.01
\]

which can directly be verified from Figure 2.


3.2 Probability bounds using linear programming

I now show how to construct the tightest possible bounds for the probability that at least $r$ out of $N$ banks default, given a set of bond prices and CDS spreads, using linear programming.

Suppose that from bond and CDS prices we obtain constraints of marginal and pairwise default probabilities of the type: $P(A_i) = a_i$, $P(A_i \cap A_j) = a_{ij}$. Then, for any $r$, we could find the tightest upper bound on $P_r$ conditional on our information set by looking for the probability system that solves the problem:

$$
\max P_r \\
\text{s.t.} \\
P(A_i) = a_i \\
\vdots \\
P(A_i \cap A_j) = a_{ij}
$$

and solve the corresponding minimization problem to find the tightest lower bound.

In general, finding a solution to problem (4) is a difficult task, as it requires us to search in the space of all possible probability systems. However, as shown by Hailperin (1965) and Kwerel (1975), probability bound problems of this type can be transformed into linear programming (LP) problems. LP problems are difficult to solve analytically, but easy to solve numerically even as the scale of the problem gets large. Additionally, the linearity of the problem guarantees that the global optimum is always achieved.

In particular, the LP approach is summarized by the following proposition, based on Hailperin (1965):\textsuperscript{12}

**Proposition 1.** The solution to problem (4) can be found as a solution to the problem:

$$
\max_p c^tp \\
\text{s.t.} \\
p \geq 0, \ i^tp = 1 \\
Ap = b
$$

\textsuperscript{12}See Appendix A for a detailed construction of the LP algorithm.
for \( c, A, b \) depending only on the available information. The lower bound is obtained by solving the corresponding minimization problem.

The following constructive proof of the result allows us to gain some intuition on the main ideas behind the linear programming approach to probability bounds, and the extent of its applicability. Start from the basic default events \( \{A_1, ..., A_N\} \), and consider the finest partition \( V \) of the sample space generated from these events through union and intersection. This partition will have \( 2^N \) elements. For example, Figure 3 reports the 8 elements of the partition for the case \( N = 3 \).

![Figure 3: Construction of the Linear Programming representation.](image)

By construction, the probability of each union or intersection of the basic events can be expressed as the sum of the probabilities of some of the events in \( V \). Since \( V \) contains \( 2^N \) elements, it is then possible to represent this probability space by a vector \( p \) with \( 2^N \) elements, each corresponding to the probability of an elementary event in \( V \).

To define uniquely the vector \( p \), we need to choose a mapping that associates each element of \( p \) with a particular event of \( V \). While this can be done in an arbitrary way, the following mapping is especially convenient. To every \( i \) from 1 to \( 2^N \) associate the binary representation of the number \((i - 1)\). Call it \( b_i \). For example, with three banks \( i = 1 \) would correspond to \( b_1 = [000] \), \( i = 2 \) to \( b_2 = [001] \), \( i = 3 \) to \( b_3 = [010] \) and so on up to \( b_8 = [111] \). Each of these vectors is a vector of indicators of one of the three basic events. For example, \( b_4 = [011] \) represents the event in which \( A_1 \) does not occur, while \( A_2 \) and \( A_3 \) occur, as is visible in Figure 3. This gives us a unique
one-to-one mapping between the set of numbers from 1 to \(2^N\) and all the elementary sets in \(V\).

Once this ordering is defined, \(p_i\), the \(i\)-th element of \(p\), can be set to equal the probability of the event represented by \(b_i\). For example, since \(b_4 = [0 \ 1 \ 1]\), we have \(p_4 = Pr\{\overline{A}_1 \cap A_2 \cap A_3\}\). This is precisely the mapping represented in Figure 3. It follows immediately that if \(A'\) is an event obtained by union or intersection of the basic default events, we will have

\[
Pr\{A'\} = c'p
\]

for some vector \(c\). For example, with 3 banks we have:

\[
P(A_1) = [0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1] \cdot p
\]

\[
P(A_1 \cap A_2) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1] \cdot p
\]

\[
P_2 = P((A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3)) = [0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1] \cdot p
\]

\[
P_3 = P(A_1 \cap A_2 \cap A_3) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \cdot p
\]

and so on. It is then easy to see how the objective function and all constraints of problem (4) can be written as linear functions of a vector \(p\) determined as described above. It is also easy to see how the only requirements for \(p\) to be a valid probability system are that \(p \geq 0\) and \(i'p = 1\), where \(i\) is a vector of ones.

### 3.3 Probability bounds with different information sets

From the derivation of Proposition 1 described in the previous section, it is easy to see that the LP approach can be used in all cases in which the objective function and all the constraints are linear in the probabilities.

This is important because, while we can obtain bond data for each bank (from which we derive \(N\) constraints on marginal probabilities), we do not have access to disaggregated CDS data containing the price of a CDS sold by every bank \(j\) against each other bank \(i, z_{ji}\). Therefore, we will not be able to include all \(P(A_i \cap A_j)\) in our information set.

Instead, as described in detail later, we only observe for each bank \(i\) the average of the CDS spread written against \(i\) across counterparties, \(z_i = \frac{1}{N-1} \sum_{j \neq i} z_{ij}\). We can then obtain, for each bank \(i\), one constraint on the average joint default probability
with its counterparties \( j \):

\[
\frac{1}{N-1} \sum_{j \neq i} P(A_i \cap A_j) = \bar{\pi}_i \ \forall i
\] (6)

where \( \bar{\pi}_i \) is a function of the observed average CDS spread \( z_i \).

These \( N \) constraints correspond to linear constraints on \( p \) in the LP problem, and can be imposed instead of the \( \frac{N(N-1)}{2} \) constraints we could impose if we observed \( z_{ji} \) for each pair \( (i, j) \). For example, with 3 banks, the first constraint would be:

\[
\frac{1}{2}[P(A_1 \cap A_2) + P(A_1 \cap A_3)] = \frac{1}{2}[0 0 0 0 0 1 1 2] \cdot p
\]

Note that if \( N = 3 \) there are only three different pairs of banks. Therefore, the three constraints on the average pairwise default risk (6) are enough to completely pin down the probabilities of each pair. However, for all \( N > 3 \), not observing all \( z_{ji} \) reduces our information set, resulting in less informative – though still valid – bounds.

The fact that the LP approach can be used with any constraint linear in probabilities also allows us to explore other interesting information sets. In particular, we can study the importance of using all available prices to construct the bounds, rather than using only the information contained in the average bond yield and CDS spread across banks, as for example was pictured in Figure 1. Proposition 1 implies that we can do that by simply averaging the available constraints across the \( N \) banks, obtaining therefore only one constraint from the average bond price and one constraint from the average CDS spread.

This exercise is useful because it allows us to gauge how much information we gain from the asymmetry of the probability system. The following Proposition holds:

**Proposition 2.** Among all probability systems with the same average marginal and pairwise default probabilities, the widest bounds on systemic risk, for any \( r \), are obtained for the symmetric system, in which all marginal probabilities are the same (and equal to the average) and all pairwise probabilities are the same (and equal to the average).

**Proof.** See Appendix B. \( \square \)

This implies that asymmetry in marginal and pairwise probabilities always results
in more informative bounds. By comparing the bounds obtained under our full information set (Section 3.1) to the ones obtained by looking at average bond and CDS spreads we can then gauge how much are we learning due to the asymmetry of the network.

To clarify the last point, we can go back to the example with three banks, and look at the bounds obtained imposing only that
\[
\frac{1}{3} \sum_i P(A_i) = 0.2 \quad \text{and} \quad \frac{1}{3} \sum_{i,j<i} P(A_i \cap A_j) = 0.05.
\]
This means that we are computing the bounds by looking across all probability systems with the same average marginal and same average pairwise probability.

The bounds we obtain in this case are:

\[
\begin{align*}
0.45 \leq P_1 & \leq 0.50 \\
0.05 \leq P_2 & \leq 0.15 \\
0 \leq P_3 & \leq 0.05
\end{align*}
\]

and they are attained by a perfectly symmetric probability system, that features

\[
\begin{align*}
P(A_i) & = 0.2 \quad \forall i \\
P(A_i \cap A_j) & = 0.05 \quad \forall i,j
\end{align*}
\]

for all banks and all pairs of banks. The gain in tightness of the bounds compared to those reported in Section 3.1 stems from the asymmetry between the pairwise default probabilities across pairs in equation 3.

### 3.4 Uniqueness of the solution

At first sight, it might seem that if a financial network is asymmetric enough, there will be a unique probability system that attains the upper bound (or the lower bound) on systemic risk. However, this is in general not the case: several probability systems exist that attain each bound. The existence of multiple solutions to the maximization problem plays an important role in section 4, where I study the contribution of different institutions to systemic risk.

If more than one probability system attains the upper (lower) bound, it is possible that some probability \(P(A')\) of an event \(A'\) is not completely pinned down at the bound. In this case, we can characterize the range of possible values for this probability by solving a second maximization/minimization problem of the type:
\[ \max_p (\min_p) v'p \]

\[ \text{s.t.} \]

\[ p \geq 0, \ i'p = 1 \]

\[ Ap = b \]

\[ c'r = c_{\text{bound}} \]

where \( v \) is the vector that corresponds to the event \( A' \) (as described in Propositions 1 and 2), and \( c_{\text{bound}} \) is the value of the upper or the lower bound on systemic risk. In other words, we look for the probability system, among those which attain the upper or the lower bound, that maximizes (minimizes) the particular probability we are interested in, \( P(A') \). Of course, if \( \max_p v'p = \min_p v'p \), that particular probability is completely pinned down at that bound on systemic risk.

In addition, I can find a “representative” probability system at the upper (lower) bound. A typical procedure employed in the LP literature is that described in Appa (2002) and reported in Appendix B. The “representative” probability system is obtained averaging different probability systems, chosen from the space of solutions to the bounds, that are as distant as possible from each other.

4 Empirical analysis

This section computes the tightest bounds on systemic risk using bond and CDS prices.

4.1 Implementation

4.1.1 Pricing models

In order to extract marginal and pairwise default probabilities from observed prices, I need to specify a pricing model for bonds and CDSs that takes into account not only default risk, but also other important determinants of prices. I use a constant hazard rate model\(^{13}\) with the following assumptions. At every time \( t \), the prices of all outstanding securities are determined assuming that at all future dates \( t + s \) the individual risk-neutral hazard rates of each firm \( i (h_{t+s}^i) \) and the joint hazard rates of

\(^{13}\)See Lando (1997), Duffie and Singleton (1999), and Hull and White (2000, 2001).
each pair $i,j$ ($h_{i,t+s}^{ij}$) will be constant and equal to some values $h_i^t$ and $h_{ij}^t$ respectively. Naturally, this is just an approximation, because prices set in this way do not take into account that at each future date $t$ these hazard rates, $h_i^t$ and $h_{ij}^t$, are going to be revised.

The model is discretized to a monthly horizon, and I assume that coupons and CDS premia are paid monthly\textsuperscript{14}. The choice of a month is motivated by the relative reference period for the double default of CDSs: I assume that a double default event occurs whenever two banks default within a month. The choice of a constant hazard rate is due to the fact that reliable CDS data is available only for 5-year fixed-maturity contracts, so there is not enough information to estimate a more flexible process for the joint hazard rate $h_{ij}^t$. I further assume that recovery rates are constant and the risk-free rate process is independent of the hazard processes under the risk-neutral probability.\textsuperscript{15}

4.1.2 Bonds

Calling $R$ the recovery rate of senior unsecured bonds, $\delta(t,T)$ the risk-free discount factor from $t$ to $T$, $c_{ik}$ the coupon of bond $k$ issued by bank $i$ and $B_{ik}$ its price, the pricing model implies:

$$B_{ik}(t,T_{ik}) = c_{ik} \left( \sum_{s=t+1}^{T_{ik}} \delta(t,s)(1-h_i^t)^{s-t}(1-\gamma_i^t)^{s-t} \right) +$$

$$+\delta(t,T_{ik})(1-h_i^t)^{T_{ik}-t}(1-\gamma_i^t)^{T_{ik}-t} + R \left( \sum_{s=t+1}^{T_{ik}} \delta(t,s)(1-h_i^t)^{s-t-1}(1-\gamma_i^t)^{s-t-1}h_i^t \right)$$ (7)

The parameter $\gamma_i^t$ is introduced in the bond pricing equation to capture the liquidity premia in bond prices. While in theory many other factors can affect the bond-CDS basis, like delivery option and restructuring clauses in CDSs, Appendix C shows that they typically have a small effect on the basis. Liquidity premia in

\textsuperscript{14}The CDS pricing formula reported below holds exactly only if CDS payments are made every month, while in reality they are made every three months. For consistency between bond and CDS pricing, the same assumption is imposed to bond coupon payments. This makes sure that the difference between bond-implied and CDS-implied probabilities (the bond/CDS basis) will not be driven by differential assumptions about the timing of cash flows of bonds and CDSs.

\textsuperscript{15}See Section 4.3 for a discussion of the robustness of the empirical results to the relaxation of many of the assumptions of the model.
bond markets, instead, have a first order effect on the bond/CDS basis (through the price of the bond) and needs to be taken into account explicitly.\footnote{For example, see Bao et al. (2008), Chen et al. (2007), Collin-Dufresne et al. (2007), Huang and Huang (2003), or Longstaff et al. (2005). Recently, Bongaerts, de Jong and Driessen (2011) have argued for the presence of generally significant liquidity premia in CDS spreads as well. However, the CDSs of the particular financial institutions considered here remained within the most traded CDSs of all even during the crisis (Fitch Ratings). As I discuss later, most results of this paper will still hold as long as the CDSs of these banks are more liquid than the bonds, which is natural given the very high margin requirements for bonds during the crisis.} As in standard models (see for example Duffie (1999)) I model this liquidity cost $\gamma_i^t$ as a per-period cost of holding the bond issued by bank $i$. $\gamma_i^t$ is assumed to be known and constant over the life of the bond at each time $t$, yet variable from $t$ to $t+1$. This parameter may be interpreted as the opportunity cost that arbitrageurs with limited capital incur when buying bonds on the margin, as in the model of Garleanu and Pedersen (2010). In a model with constant hazard rates, the liquidity premium that arises in the setting of Garleanu and Pedersen (2010) corresponds to the term $\gamma_i^t$ in the bond pricing equations I employ.

Given that $\gamma_i^t$ is unobservable, estimating it directly can be quite difficult. The methodology presented in this paper, however, allows me to construct bounds on systemic risk even without a full estimation of this process. Imposing a lower bound – a much easier task – is sufficient. By imposing lower bound for $\gamma_i^t$ (call it $\gamma_i^{\ell}$) we obtain an upper bound on $P(A_i)$ for each bank. This upper bound corresponds to the probability of default implied by the bond pricing formula assuming $\gamma_i^t = \gamma_i^{\ell}$. Call this value $h_i(\gamma_i^{\ell})$. We can then modify the maximization problem (eq. 4) to compute the bounds by replacing the equality constraints $P(A_i) = \alpha_i$ with inequality constraints of the form

$$P(A_i) \leq h_i(\gamma_i^{\ell})$$

(8)

I examine three plausible bounds on $\gamma_i^t$. The weakest possible assumption is just to assume that $\gamma_i^t \geq 0$, for all $t$ and $i$: a large literature has established that bond liquidity premia are surely not negative.

A second possibility is to assume that liquidity premia were, during the crisis, at least as high as they were in 2004 (the beginning of my sample). To implement this lower bound, start by assuming that $\gamma_i^t$ can be decomposed into the product of two components: $\gamma_i^t = \alpha_i \lambda_i$. $\alpha_i$ is fixed over time but varies by firm and is scaled to capture the average liquidity premium of bank $i$ in 2004 (equivalently, $\lambda_i = 1$ in
2004). $\lambda_i$ captures the common movement in liquidity premia for financial firms. If we believe that counterparty risk played a minor role in CDS pricing back in 2004, we can estimate $\alpha_i$ directly from the average bond/CDS basis in 2004\(^{17}\), and impose $\gamma_i \geq \alpha_i$.

Finally, my preferred approach obtains a time-varying lower bound for the liquidity process, $\gamma_i$, by comparing the financial institutions in the sample to non-financial institutions with high credit rating and therefore with the lowest margins and cost of funding. A CDS written by a financial institution on a safe non-financial firm, $j$, is much less likely to be affected by the risk of double default, so the CDS spread only depends on the individual hazard rate $h_i^J$. The bond/CDS basis then reflects only liquidity premia. Under this assumption, I proceed as follows. For a set $J$ of nonfinancial firms with high credit rating, I estimate $\gamma_i^J$ from the observed bond/CDS basis. In particular, for each $t$ I find the value of $\gamma_i^J$ that makes the hazard rate $h_i^J$ implied from bonds exactly price the CDS spread of bank $j$, therefore explaining the whole bond-CDS basis. I then decompose $\gamma_i^J$ as $\gamma_i^J = \alpha_i \lambda_i^*$, and extract the common component for non-financial firms $\lambda_i^*$, again normalized so that $\lambda_i^* = 1$ in 2004\(^{18}\). The margin requirements and other liquidity-related costs for the bonds of these firms arguably increased less during the crisis than for bonds issued by financial firms, so $\lambda_i^* \leq \lambda_i$. I then obtain a third possible constraint on the liquidity process of financial firms: $\gamma_i \geq \alpha_i \lambda_i^*$. Note that if liquidity premia increase in a way that is correlated with systemic risk, this approach will capture it and allow us to obtain tighter bounds.

Another advantage of imposing only a lower bound on $\gamma_i$ is that most results go through if we instead interpret $\gamma_i$ as the relative liquidity of bonds and CDSs. While credit default swaps are more liquid than bonds, they nonetheless incorporate some liquidity premia. As long as the assumptions on the lower bound for $\gamma_i$ are valid when interpreted in terms of relative liquidity (for example: $\gamma_i^J \geq 0$ corresponds to bonds being always less liquid than CDSs, and so on), the bounds computed here will be valid.

In some cases, after removing the component of the bond yield attributed to

\(^{17}\)I take the average of the basis in 2004 as opposed to the basis on, say, 1/1/2004, to reduce noise. The basis was not volatile during that period, so the exact window used to define $\alpha_i$ makes little difference to the results. It also makes very little difference if one uses the basis in any other period before mid 2007.

\(^{18}\)This can be done under the assumption that $\gamma_i^J$ is observed with independent proportional noise $\epsilon_i^J$, i.e. we observe $\tilde{\gamma}_i^J = \gamma_i^J \epsilon_i^J$; we can then estimate the series $\lambda_i^*$ for each $t$ using OLS on the logs.
liquidity, the remaining part of the yield spread can be lower than the CDS spread. For example, when the liquidity process is set so that the average basis in 2004 is zero, in 2004 half of the banks will have a positive liquidity-adjusted basis. In these cases I reduce the effect of liquidity to the point where bond-implied probabilities are as high as the CDS-implied probabilities, and counterparty risk goes to zero. This phenomenon occurs less frequently as the financial crisis unfolds and the basis widens for more banks.

4.1.3 CDSs

The pricing model for CDSs is obtained under assumptions similar to those of the bond pricing model. The price of a CDS depends additionally on the payoff in case of joint default events of the reference entity $i$ and the counterparty $j$ in the following way. Conditional on not having had a credit event up to time $t + s$, and therefore the contract still being active, if the seller does not default within the next month but the reference entity defaults, the payment is made in full. Therefore, a month is considered an amount of time sufficient for the seller to establish whether to default on the CDS obligation given that the reference entity defaulted. If the seller defaults within the next month but the reference entity does not, the contract terminates with a zero value: the buyer can just replace the contract with some other counterparty at the same price it was paying. \(^{19}\) If both the seller and the reference entity default in the same month, I assume that the two defaults happen in a connected way and only an amount $S$ of the full payment is recovered. This case corresponds to the double default case discussed in section 2. Assuming $S$ is constant over time, the price of the CDS sold by bank $j$ against the default of $i$ will solve (see Appendix C for further details):

$$
\sum_{s=t}^{T-1} \delta(t, s)(1 - h_t^i - h_t^j + h_t^{ij})^{s-t} z_{ji} = \quad (9)
$$

\(^{19}\)This is consistent with the assumption of constant hazard rates, in which the credit risk of the reference entity is constant over time as long as it does has not defaulted. The pricing formula remains approximately the same as long as the hazard rate of default of the reference entity, conditional on surviving until the next month, remains of the same order of magnitude as it was before the default of the counterparty. In this case, the effect on the price of the CDS is of the order of magnitude of the square of the CDS spread, which is very small.
\[
\sum_{s=t+1}^{T} \delta(t, s)(1 - h^i_t - h^j_t + h^{ij}_t) s^{-t-1} \left\{ \left[ h^i_t - h^{ij}_t \right] (1 - R) + h^{ij}_t S(1 - R) \right\}
\]

The left-hand side of the formula represents the present value of payments to the protection seller; they only occur as long as neither a credit event occurred nor the counterparty defaulted. The right-hand side represents the expected payment in case of default. In each period, conditional on both firms surviving until then, there is a probability \(h^i_t - h^{ij}_t\) that the reference entity defaults while the counterparty has not defaulted, so that the payment of \((1 - R)\) is made in full. With probability \(h^{ij}_t\) there is a double-default event, and only a fraction \(S\) of that payment is recovered. If only the counterparty defaults the contract ends with zero value due to the assumption of constant hazard rates.

To back out probabilities from bonds and CDS prices, we need to calibrate the parameters \(R\) and \(S\). As discussed in Section 2, due to the status of CDSs in bankruptcy \(S\) is at least as large as \(R\). As a baseline case, I assume \(S = R = 30\%\), which corresponds to the case in which little or no collateral is posted on the CDS contract. Section 4.3 explores robustness of the results to different levels of \(R\) and \(S\), as well as the case of stochastic recovery rates correlated with the default events. In particular, it discusses why all the results hold strongly for values of \(S\) as large as \(90\%\).

Using a linear approximation derived and discussed in Appendix C, it is possible to rewrite the CDS spread as:

\[
z^i_{ji} = (h^i_t - (1 - S)h^{ij}_t) \frac{\sum_{s=t+1}^{T} \delta(t, s)}{\sum_{s=t}^{T-1} \delta(t, s)} (1 - R)
\]

This representation is linear in the event probabilities and could then be imposed as a constraint in the LP problem.

However, remember that we only observe an average of the quotes provided by the \(N - 1\) counterparties, \(\bar{z}_i = \frac{1}{N-1} \sum_{j \neq i} z_{ji}\), for each bank \(i\). We can then compute the averaged version of equation (10) across \(j\):

\[
\bar{z}^i_t = \left\{ h^i_t - (1 - S) \left[ \frac{1}{N-1} \sum_{j \neq i} h^{ij}_t \right] \right\} \frac{\sum_{s=t+1}^{T} \delta(t, s)}{\sum_{s=t}^{T-1} \delta(t, s)} (1 - R)
\]

Therefore, the results will depend only on the average counterparty risk across coun-
Finally, I do not observe the exact set of counterparties that contribute quotes each day. Therefore, I compute bounds for the group of 15 largest dealers by volume and trade count, which are likely to represent the sample of firms from which the quotes come from\textsuperscript{21}. I assume that each of these large dealers has the same probability of contributing a quote, and I explore alternative hypotheses in section 4.3.

### 4.1.4 Data and estimation method

The data cover the period from January 2004 to June 2010 with daily frequency. For each of the 15 institutions considered, I obtain clean closing prices\textsuperscript{22} from Bloomberg for senior unsecured zero and fixed coupon bonds with maturity less than 10 years. Given that the maturity of CDSs is 5 years, it would also be possible to use only outstanding bonds of remaining maturity close to 5 years when comparing bonds and CDSs. However, for many European firms we do not have enough bonds around the 5-year maturity for all periods, so that we need to use a wider window.\textsuperscript{23} The quotes provided by Bloomberg are indicative, not necessarily actionable. However, if the bond is TRACE-eligible, Bloomberg reports the closing price from TRACE, which corresponds to an actual trade\textsuperscript{24}. I exclude callable, putable, sinkable, and structured bonds, since their prices reflect the value of the embedded options. I remove all bonds for which I have price information for less than 5 trading days.

\textsuperscript{20}Averaging makes the results robust to the possibility that the cross-sectional dispersion of quotes might not fully reflect the cross-sectional dispersion in joint default risk with the reference entity, as suggested under the assumptions discussed in Section 2 by Gandhi et al. (2011). Under similar assumptions, Bai and Collin-Dufresne (2011) provide evidence that average counterparty risk is priced.

\textsuperscript{21}The market is extremely concentrated, and the top 10 dealers account for more than 90% of the volume of CDS sold.

\textsuperscript{22}Clean prices adjust for the coupon accrued between actual coupon payment dates and therefore are most appropriate to estimate equations (7) and (10), in which coupon payments and CDS premia are assumed to be made monthly, and not every 6 or 3 months respectively. The exact timing of coupon and CDS payments has a small effect on the implied hazard rates of bonds and CDSs once clean prices are used.

\textsuperscript{23}Similar results obtain using windows of 4 to 6 years, and 2 to 8 years.

\textsuperscript{24}Section 4.3 and Appendix D discuss robustness result based only on TRACE prices of large transactions.
Table 1

<table>
<thead>
<tr>
<th>Institution</th>
<th>Avg valid bonds</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abn Amro</td>
<td>3.8</td>
<td>1.8</td>
<td>2.2</td>
<td>4.0</td>
<td>4.4</td>
<td>3.3</td>
<td>5.0</td>
<td>7.3</td>
</tr>
<tr>
<td>Bank of America</td>
<td>32.6</td>
<td>17.6</td>
<td>25.5</td>
<td>29.4</td>
<td>32.9</td>
<td>35.7</td>
<td>42.8</td>
<td>56.8</td>
</tr>
<tr>
<td>Barclays</td>
<td>15.7</td>
<td>3.1</td>
<td>3.0</td>
<td>2.4</td>
<td>2.5</td>
<td>9.5</td>
<td>42.0</td>
<td>82.0</td>
</tr>
<tr>
<td>Bear Stearns</td>
<td>11.7</td>
<td>7.2</td>
<td>9.8</td>
<td>13.5</td>
<td>15.5</td>
<td>15.6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Bnp Paribas</td>
<td>7.6</td>
<td>0.5</td>
<td>2.0</td>
<td>3.6</td>
<td>4.5</td>
<td>7.1</td>
<td>19.9</td>
<td>24.6</td>
</tr>
<tr>
<td>Citigroup</td>
<td>36.5</td>
<td>21.6</td>
<td>24.3</td>
<td>31.7</td>
<td>40.0</td>
<td>43.2</td>
<td>49.5</td>
<td>54.5</td>
</tr>
<tr>
<td>Credit Suisse</td>
<td>5.4</td>
<td>1.9</td>
<td>2.3</td>
<td>2.8</td>
<td>2.7</td>
<td>5.0</td>
<td>11.6</td>
<td>17.4</td>
</tr>
<tr>
<td>Deutsche Bank</td>
<td>42.8</td>
<td>5.3</td>
<td>10.4</td>
<td>42.3</td>
<td>68.9</td>
<td>54.6</td>
<td>61.8</td>
<td>70.9</td>
</tr>
<tr>
<td>Goldman Sachs</td>
<td>42.4</td>
<td>19.3</td>
<td>26.1</td>
<td>34.3</td>
<td>40.4</td>
<td>52.5</td>
<td>67.0</td>
<td>72.9</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>20.4</td>
<td>6.6</td>
<td>11.1</td>
<td>14.0</td>
<td>17.4</td>
<td>25.9</td>
<td>38.4</td>
<td>38.9</td>
</tr>
<tr>
<td>Lehman Brothers</td>
<td>20.1</td>
<td>10.5</td>
<td>15.2</td>
<td>20.5</td>
<td>26.5</td>
<td>31.4</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Merrill Lynch</td>
<td>37.7</td>
<td>23.6</td>
<td>33.8</td>
<td>41.5</td>
<td>45.8</td>
<td>46.3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Morgan Stanley</td>
<td>28.4</td>
<td>12.5</td>
<td>14.6</td>
<td>17.5</td>
<td>22.2</td>
<td>33.5</td>
<td>55.0</td>
<td>59.0</td>
</tr>
<tr>
<td>UBS</td>
<td>9.1</td>
<td>0.3</td>
<td>0.7</td>
<td>1.0</td>
<td>3.1</td>
<td>8.9</td>
<td>25.5</td>
<td>40.6</td>
</tr>
<tr>
<td>Wachovia</td>
<td>6.1</td>
<td>2.9</td>
<td>3.5</td>
<td>5.7</td>
<td>7.4</td>
<td>9.1</td>
<td>7.7</td>
<td>7.3</td>
</tr>
</tbody>
</table>

Note: first column reports average number of bonds for each institution that are used for the estimation of marginal default probabilities. Columns 2-8 break this number down by year.

I consider only bonds denominated in five main currencies: USD, Euro, GBP, Yen, CHF. Since Bloomberg data on European bonds is fairly limited, I integrate them with bond pricing data from Markit whenever it adds at least 5 observations to the price series of each bond. As the reference risk-free rate, I use government zero-coupon yields obtained from Bloomberg.\textsuperscript{25} As discussed in section 4.3, results are robust to using swap rates.

Table 1 reports some statistics on the availability of bond data. For each institution, we can see the average daily number of valid bond prices available for the estimation, in total and by year. For example, the default probability for Bank of America $h_i^t$ is estimated using on average 32.6 bonds each day. The Table shows that for some European dealers, bond data is scarce especially in the early part of the sample.

\textsuperscript{25}I instead bootstrap the Swiss yield curve from the government coupon yield curve, using linear interpolation and assuming bonds trade at par.
Turning to CDS data, I obtain data on the 5-year CDS contract (the only liquid maturity throughout the sample period) from Markit.26 Markit reports spreads that are obtained by averaging the quotes reported by different dealers, after removing stale prices and outliers.27 Table 2 reports summary statistics on CDS spreads. While CDS spreads between 2004 and 2010 are usually quite low, on the order of 50bp (0.05%), they reach levels higher than 1000bp (10%) in some periods. On the right side of Table 2, I report statistics for the basis $z_i - (y_i - r^F)$, where $z_i$ is the CDS spread and $y_i$ is the bond yield, computed using the interpolated 5-year bond yield and the 5-year Treasury rate. As expected, the basis is usually negative, because the CDS spread is lower than the corresponding bond yield spread. In a few cases, however, the basis becomes positive - which, as I explain in Section 4.1, I denote as zero counterparty risk for that day. Since this phenomenon typically occurs for only a few days, it will not affect the general behavior of the bounds.

Using these data, together with a calibration of the lower bound for the liquidity process $\gamma^i_t$, I estimate the risk-neutral default probability $h^i_t(\gamma^i_t)$ in each trading day $t$. The estimation is performed separately for each firm, using the cross section of

26All results are robust to using Bloomberg - CMA data instead.
27Removing outliers, while helpful to reduce noise coming from erroneous prices, can potentially bias the reported CDS spread away from the average spread if the distribution of quotes is skewed. This in particular can be a concern if the distribution of quotes is left-skewed, i.e. most dealers have low counterparty risk but a few dealers have higher counterparty risk, as the observed CDS spread would be biased upwards. Note also that while we only observe mid quotes, theoretically ask quotes (at which banks sell CDSs) should be used for this exercise. Using mid quotes might lead us to overestimate the basis, which would result in a wider upper bound on systemic risk. Therefore, the results will still be valid; the upper bound would be less tight than it could otherwise.
bonds issued by firm $i$ which are outstanding at time $t$. In particular, for each firm $i$ and time $t$ I find the value of the hazard rate $h^i_t(\gamma^i_t)$ that minimizes the average absolute pricing error of equation (7) across outstanding bonds issued by $i$ under the assumption $\gamma^i_t = \gamma^i_t$. I use absolute pricing errors to reduce the impact of outliers, but all the results are robust to the use of squared pricing error as a loss function. This procedure allows me to obtain for each firm an upper bound on the marginal default probability: $P(A_i) \leq h^i_t(\gamma^i_t)$, obtaining a first set of constraints in the optimal bounds problem.

As discussed in Section 3, a second set of constraints in the maximization problem comes from the information about joint default risk contained in CDS spreads. In theory, given $h^i_t$ and the observed $z^i_t$, we could estimate $\frac{1}{N-1} \sum_{j \neq i} h^{ij}_t$ from the CDS spread against bank $i$ using equation (11). Since $h^{ij}_t$ is the monthly probability of joint default of $i$ and $j$, it corresponds exactly to $P(A_i \cap A_j)$ when the reference horizon is one month. Therefore, we could estimate $\frac{1}{N-1} \sum_{j \neq i} h^{ij}_t$ and then impose as a constraint, for each $i$,

$$\frac{1}{N-1} \sum_{j \neq i} P(A_i \cap A_j) = \frac{1}{N-1} \sum_{j \neq i} h^{ij}_t$$

Remember however that from bond prices we do not obtain $h^i_t$, but only an upper bound on it, so we cannot estimate $\frac{1}{N-1} \sum_{j \neq i} h^{ij}_t$ from CDS spreads. Instead, we can simply impose equation (11) directly in the maximization problem, as a constraint linear in the unobserved marginal and pairwise probabilities.

### 4.2 Empirical Results

#### 4.2.1 Bounds on systemic risk

I start by presenting the empirical results under my preferred liquidity assumption, that calibrates the liquidity process using the bond/CDS basis of nonfinancial firms. In the notation of Section 4.1.2, I assume $\gamma^i_t \geq \alpha^i_t \lambda^*_t$, where $\lambda^*_t$ is a time-varying component estimated from nonfinancial firms that allows to capture at least some of the time variation in liquidity premia over the crisis.

Figure 4 presents the bounds on the probability that at least $r$ financial institutions default within a month, $P_r$, for $r$ between 1 and 4. The upper and lower bounds on the probability that at least one bank defaults vary significantly over time, and the
bounds are quite informative. The width of the bounds is less than 1% before 2008, and increases to about 3% in 2009. For all \( r > 1 \), the lower bound is 0, but the upper bound is relatively tight, and displays noticeable time variation between 2007 and 2010. For example, the probability that at least 4 banks default is at most a few basis points before March 2008, and rises up to about 1% at the peak in 2009.

![Figure 4: Upper and lower bounds on systemic events under the assumption that liquidity premia of bonds issued by financial firms increased during the crisis at least as much as those of nonfinancial firms. \( P_r \) is the monthly probability of at least \( r \) banks defaulting, for \( r = 1, 2, 3, 4 \). Bounds are smoothed with a 3-day moving average.](image)

Taken together, the bounds provide a description of the financial crisis consistent with standard measures of systemic risk. All the bounds in the Figure suggest an
increase in systemic risk up to early 2009, followed by a decrease starting in May 2009 after the relatively comforting results from the stress tests on these banks. Systemic risk picks up again at the very end of the sample (June 2010), following worries about the stability of the European banking system.

While the bounds on different degrees of systemic risk – from $P_1$ in the top panel to $P_4$ in the bottom panel – often move in similar ways during the crisis, significant differences emerge during specific periods. In particular, before Bear’s collapse in March 2008, the probability that at least one bank defaults increases noticeably but this spike does not appear for the probability that many banks default (bottom three panels): the probability of systemic events of more than two banks is very close to zero up to the day Bear Stearns’ collapses. Similarly, during the month after Lehman’s collapse in September 2008 we observe a spike in systemic risk, but the spike is smaller for $r > 1$ than it is for $r = 1$. For these periods, the bounds suggest an interesting decomposition of the movements of bond yields and CDS spreads into idiosyncratic and systemic risk: systemic risk was not spiking as much as idiosyncratic risk was (as captured by $P_1$).

The results in Figure 4 are obtained under the most stringent calibration of the liquidity process among the three presented in Section 4.1.2. However, it is interesting to note that the main results of the analysis, and in particular the decomposition between idiosyncratic and systemic risk during some episodes are present even under much weaker assumptions about $\gamma$. Figure 5 reports the bounds obtained under the three different liquidity calibrations discussed before. The top panel reports, for reference, the bounds on $P_1$ under the preferred liquidity calibration, the same as in Figure 4. The bottom three panels report the bounds on $P_4$ under the three liquidity calibrations discussed in Section 4.1.2: nonnegative liquidity premia, liquidity premia at least as high as 2004, and liquidity premia calibrated to nonfinancials.

Comparing the bottom three panels we can see that while tightening liquidity does reduce significantly the upper bound on $P_4$, the main decomposition of idiosyncratic and systemic risk is evident even under the weakest calibration $\gamma_i^l \geq 0$ (second panel of the Figure). The decomposition holds very strongly in the months around Bear Stearns (in which none of the bottom three panels show a peak similar to the one we observe for $P_1$ in the top panel), and a similar result holds, less strongly, for the month after Lehman’s collapse.
Figure 5: Bounds on the monthly probability of at least 1 bank (top panel) and 4 banks (bottom three panels) defaulting under different liquidity assumptions. The top panel shows $P_1$ with liquidity calibrated to nonfinancials. The bottom three panels show $P_4$ under the three liquidity assumptions: nonnegative, at least as high as 2004, and calibrated to nonfinancials.

The optimal bounds on systemic risk give the following account of the financial crisis. Up to the collapse of Bear Stearns, bond and CDS prices indicate that systemic risk was low. The upper bound on $P_4$ does not indicate a sharp increase in systemic risk at the beginning of 2008, contrary to what most other measures of systemic risk suggest. As confirmed by the top panel of Figure 4, that increase is due to idiosyncratic, not systemic risk. After jumping in March 2008, systemic risk increased smoothly up to April 2009. After Lehman’s collapse in September 2008,
the probability that at least one (other) bank would default shows a large spike for a whole month. However, a smaller spike is observed for the probability that many banks default. The large spike observed in CDS spreads and bond yields in September and October 2008 then corresponds more to an increase in idiosyncratic default risk than in systemic risk. Systemic risk then declines in 2009 and 2010.

It is useful to note that the main patterns of this decomposition can be traced back to the raw data depicted in Figure 1. Around these episodes, the bond yields and CDS spreads tend to spike but the difference between the two (the bond/CDS basis) does not. Since the methodology presented in this paper extracts information on counterparty risk, and therefore pairwise default risk, from the bond/CDS basis, periods in which bond yields and CDS spreads spike but the basis does not cannot be interpreted as episodes of high systemic risk. If agents really were worried about the joint default risk of these banks they should have required a much higher discount for these CDSs than we observe, since these were exactly the banks that were selling protection against each other. Additionally, this effect is even stronger once we take into account that part of the basis is due to liquidity, not counterparty risk. The methodology presented in this paper allows us to capture and aggregate in the best possible way this information to obtain specific bounds for systemic risk events of different severity.

These empirical results are also consistent with the common view of the events occurred during the financial crisis. Before Bear Stearns collapsed, market participants were aware of the possibility that banks could fail. However, a joint default event of multiple banks within a short horizon was seen as unlikely, and therefore counterparty risk for a buyer of CDS protection was perceived to be low. Bear’s collapse showed that defaults of these large banks could happen suddenly, in a way that would not allow buyers to cover their counterparty exposures in time. Only then the basis starts to widen. Similarly, while people observed Lehman’s sudden default in September 2008, they also observed the government saving Merrill Lynch and AIG in the next two days - thus avoiding a multiple default event. Markets learned that the government might let a bank fail but was unlikely to let many banks fail - hence the larger spike in $P_1$ than in $P_2, P_3$ and $P_4$.

All these results were derived for risk-neutral probabilities. But importantly, if this decomposition between idiosyncratic and systemic risk holds for risk-neutral probabilities, it should hold even more strongly for objective probabilities. Around Bear’s
collapse and after Lehman defaults, we observe that $P_1$ spikes, but $P_2$, $P_3$ and $P_4$ do not increase as much. Suppose $P_1$ jumps due to an increase in risk premia: agents increase their expectations of marginal utility in states of the world when at least one bank defaults. Since events in which *many* banks default arguably happen in states of the world with even higher marginal utility, we would then expect $P_2$, $P_3$ and $P_4$ to increase even more. But empirically, the latter do not increase as much in these cases. Therefore these episodes must be driven by movements in the objective probabilities, and not in risk premia: the objective probability that one bank would fail increases while the objective probability that many default does not.

### 4.2.2 Bounds under smaller information sets

To better understand the importance of using *all* available information aggregated with the methodology introduced in this paper, we can compare the optimal bounds with bounds obtained under smaller information sets.

![Figure 6: Bounds on $P_4$ (probability that at least 4 banks default) under different information sets.](image)

The top panel of Figure 6 compares the bounds on $P_4$ obtained using all information available to bounds obtained using *only* bonds prices or *only* CDS spreads. In particular, the thin lines in the Figure represent the upper bounds obtained using
only bond prices, i.e. discarding the constraints coming from CDS prices (all lower bounds are 0). The dotted lines use only CDS data, ignoring the constraints coming from bond prices (upper bounds on marginal probabilities). Both sets of bounds ignore the information contained in the bond/CDS basis. The shaded bounds represent the full-information bounds, with liquidity calibrated to nonfinancials.\footnote{When we compute the bounds using only bond information, we cannot adjust for liquidity by using information in the bond/CDS basis. Therefore, I use here the assumption $\gamma_i > 0$ when computing the bonds-only bounds. The bonds-only bounds in the Figure look very similar if we instead compute them adjusting the bond prices with the liquidity process calibrated to non-financial firms, therefore using some information from the bond/CDS basis.}

The bounds on $P_4$ that do not use information contained in the basis tell quite a different story than the bounds that use all information available. In particular, they present a sharp increase in systemic risk before March 2008 and a much larger spike after September 2008. In fact, these bounds on $P_4$ closely resemble the behavior of the bounds on idiosyncratic risk $P_1$ shown in the top panel of Figure 4. They do not allow to distinguish relative movements of idiosyncratic and systemic risk. This confirms the importance of considering the information in the bond/CDS basis to learn the most about systemic risk.

The bottom panel of Figure 6 compares instead the optimal bounds to bounds obtained using only information on the average bond yield and average CDS spread across banks. As discussed in section 3, bounds obtained using only average information coincide with bounds obtained in a fully symmetric system with all marginal default probabilities equal across banks and all pairwise probabilities equal across pairs. Therefore, comparing these bounds with the full information bounds reveals how much we learn from the asymmetric structure of the information we have. The larger the cross-sectional difference in marginal probabilities across banks, and the larger the difference in joint default probabilities across pairs, the tighter the bounds will be. The last panel of Figure 5 shows that at least in some episodes (like around Lehman’s default) considering the full information set is crucial to distinguish periods with high idiosyncratic risk and periods with high systemic risk – the asymmetry of the network therefore contains valuable information for measuring systemic risk.

4.2.3 Individual contributions to systemic risk

The method described here also allows us to study the evolution of the default risk of each bank and its relation with the rest of the network. In particular, I solve for
the probability systems that attain the upper bound for \( P_4 \), the probability that at least 4 banks default, and I study the configuration of the financial network in the scenarios of highest systemic risk.\(^{29}\)

In general, several probability systems attain the upper (and lower) bounds on systemic risk. I follow the procedure described in section 3 and Appendix B to numerically solve for a “representative” probability system in the space of solutions. This system offers an insight about the typical configuration of the network at the bounds. For the pairwise default probabilities, I also report the maximum and minimum values that can be achieved within the space of solutions whenever these are not completely pinned down at the bounds. Marginal probabilities are pinned down at the bound.

Figure 7: Marginal and pairwise average monthly default probabilities for part of the network in the high systemic risk scenario (max \( P_4 \)) as of 08/06/2008, with the liquidity process calibrated to that of nonfinancial firms.

Figure 7 reports a partial snapshot of the network as of August 6th 2008, five weeks before Lehman’s collapse. The nodes of the diagram are associated with the

\(^{29}\)In this section I look at the bounds for \( P_4 \) obtained by calibrating the liquidity process to that of nonfinancial firms.
individual banks and present monthly marginal probabilities of default. The segments that connect the nodes report the joint default probability of the two intermediaries. The Figure reports the estimate of pairwise default probability in the “representative” solution, with the range of maximum and minimum possible values in parentheses.

The figure implies that the pair at highest risk of joint default is Merrill Lynch with Lehman Brothers, followed by the pair of Lehman Brothers and Citigroup. The prices of bonds and CDSs were consistent with a high joint default risk of Lehman and Merrill even 5 weeks before the weekend in which both collapsed (September 13-14, 2008). Other segments of the graph show considerable heterogeneity in the marginal probabilities of default, but especially so in the pairwise probabilities, which are particularly informative about the structure of the network. I omit from the graph several banks for which the joint default risk with other banks is close to zero, even though their marginal default risk is relatively high – which is consistent with their defaults being approximately independent from the other banks. This result shows that $P(A_i)$ and $P(A_i \cap A_j)$ can move independently of each other, as they respond to different economic forces.

Using a similar approach, for each pair of banks $i$ and $j$ I track the evolution of $P(A_i), P(A_j)$ and $P(A_i \cap A_j)$ over time. Figure 8 plots a 3-day moving average of these probabilities for three different pairs (all combinations of Lehman, Merrill Lynch, and Citigroup) in the representative solution. The upper panel reports the marginal probabilities, and the lower panel reports the joint probabilities. These graphs confirm the relatively high degree of heterogeneity and variability in marginal default probabilities across banks, but even more so for joint probabilities; joint probabilities can behave quite differently than marginal probabilities. The data confirm that the markets anticipated the possibility of joint collapse of Lehman Brothers and Merrill Lynch for the two months prior to that event, while the probability of joint default of Lehman and Citigroup’s increases in the last few weeks.

We can now turn to study how each institution contributed to systemic risk. One way to capture this is to compute the probability that institution $i$ is involved in a multiple default event, $Pr\{\text{at least 4 default } \cap i \text{ defaults}\}$. By applying the techniques described in section 3, I verify that this probability is uniquely identified.

---

30 In this day: the European banks and Goldman Sachs.
31 If defaults are independent, the effect of counterparty risk on the basis is on the order of magnitude the square of the default probabilities, which is extremely small.
Figure 8: Marginal and pairwise default probabilities over time for selected banks under the liquidity process calibration to nonfinancial firms.

Figure 9: Individual contributions to systemic risk for selected banks, under the calibration of the liquidity process to that of nonfinancial firms, at the upper bound for the probability that at least 4 banks default.
at the upper bound on systemic risk. Figure 9 plots this contribution for four banks (Citigroup, Lehman Brothers, Merrill Lynch and Bank of America) as well as the average contribution across the other banks.\textsuperscript{32} The Figure shows large heterogeneity across dealers, both in levels and in changes. While the contribution to systemic risk increases for all banks after August 2007, the growth is faster for Lehman, Merrill Lynch and Citigroup than for the other banks. Lehman Brothers appears the most systemic institution at almost all times since August 2007, and particularly so several months before its default. After September 2008, Citigroup and Bank of America become the most systemic institutions.

4.3 Robustness

The empirical results described in section 4.2 are obtained imposing assumptions on recovery rates and pricing models. In this section I briefly discuss how the results depend on these assumptions, and I report the full derivation of the robustness tests – as well as their empirical implementation – in Appendix D.

With respect to recovery rates, equations (1) and (2) imply that changes in the recovery rate of bonds, $R$, scale the implied default probabilities from all prices in the same direction. Therefore, such changes scale the level of the bounds uniformly for the whole period, without affecting the main empirical conclusions about the evolution of systemic risk. In Appendix D, besides reporting results for different levels of $R$, I also show that the main conclusions hold if $R$ decreases during periods of high distress, as well as if $R$ follows a simple stochastic process that features lower recovery rates if several banks default.

More interesting is the robustness with respect to the recovery rate in case of double default, $S$. Appendix D shows that the results are robust to changes in this parameter (at least up to a recovery rate of 90\%).\textsuperscript{33} The effect of changes in $S$ depends crucially on the liquidity-adjusted bond/CDS basis of each bank. For some banks, the basis is small enough that it can be completely explained by counterparty risk. For these banks, an increase in $S$ means that the same basis can account for higher counterparty risk. For other banks, instead, the basis is large enough that it cannot be completely explained by counterparty risk: a part of the basis must be

\textsuperscript{32}To improve readability, I plot a two week moving average.\textsuperscript{33} For $S$ equal to 100\% counterparty risk does not matter for the price of a CDS, and the information about joint default risk disappears.
explained by liquidity. For these banks, as $S$ increases, the maximum joint default risk with other banks must decrease. The robustness of the results to changes in $S$ stems precisely from the fact that the effect of changes in $S$ is different and opposite across banks with large and small basis, so that the increase in systemic risk due to one bank is offset by a decrease in risk due to another.

Appendix D also discusses the robustness of the results under an alternative pricing model for bonds and CDSs, that features a more flexible specification for the hazard rate process of default. This specification allows us to capture periods in which the hazard rate is very high at short horizons but reverts to lower levels in the long run. In this case, the level of the bounds at the one month horizon increases, but its evolution over time remains the same, thus confirming the main results presented above.

The Appendix also shows that the empirical results still hold when the swap rate is used as the risk-free rate, when I restrict the sample to US banks only, and when I only use bond data from large TRACE transactions. Finally, I show what assumptions are needed to be able to estimate systemic risk when bonds and CDSs expressed in different currencies are used together. I also show robustness to different assumptions about the weighting of pairwise default probabilities when representing the average CDS spread across counterparties. The latter also accounts for perturbations of the assumptions about the heterogeneity of recovery rates across counterparties.

5 Conclusion

I study the role of counterparty risk in CDS markets for the measurement of systemic risk. Because counterparty risk affects the price of a CDS but not the underlying bond, by combining the information from bonds and CDS spreads we can learn about the joint default risk of pairs of financial institutions.

Learning about pairs of institutions, however, is not sufficient to pin down systemic risk. I introduce a methodology that allows us to characterize the risk of joint default events without any assumptions about the joint distribution function. In particular, using linear programming we can construct the tightest bounds on the probability that many banks default conditional on the information we can extract directly from bond and CDS prices.

The empirical analysis of Section 4 computes the bounds on the probability that
at least $r$ banks default in the same month, for $r = 1, 2, 3, 4$, during the financial crisis. By combining optimally the information contained in bonds and CDS spreads we learn before March 2008 systemic risk (the probability that two or more banks default together) was consistently low. At the same time, the idiosyncratic risk of one bank defaulting started increasing since August 2007 and saw a large spike in early 2008. Only after Bear Stearns’ collapse in March 2008 we see an increase in systemic risk as well. Systemic risk then keeps increasing over time, up to March 2009, when it starts to decline. In September 2008, after Lehman goes bankrupt, the probability that at least 1 bank defaults shows a large spike, but the probability that many bank defaults (at least $r$, for $r = 2, 3, 4$) shows a smaller spike. Just like for March 2008, for the month after Lehman’s collapse we obtain an interesting decomposition of the movements in bond and CDS spread between changes in the risk that one bank would default and the risk that many banks default – systemic default risk. The analysis also shows that these results cannot be obtained without using all the information available to us, all bond prices and all CDS prices.

This approach also allows us to pin down how much each bank contributes to systemic risk at the bounds. It shows that months before the weekend in which Lehman Brothers and Merrill Lynch collapsed, the probability of joint default of the two was estimated to be much higher than any other pair. It also shows that Lehman Brothers was consistently indicated as the most systemic institution since at least 6 months before its default.

These bounds on systemic risk do not aim to represent the definitive measure of systemic risk. They depend on several modeling assumptions and they are affected by limitations in the availability of the data. However, they can be useful to complement other methodologies, by producing restrictions that need to be satisfied in order for any measure to be consistent with observed data. In addition, the methodology enables us to obtain informative bounds while imposing minimal assumptions about elements of the markets, like bond liquidity, of which our knowledge is limited. As our understanding of these elements improves, tighter constraints will result in tighter bounds. The methodology can also be easily expanded to the use of new and better data on bond prices, CDS spreads, and other instruments, which enables further tightening of the bounds.

Finally, these bounds reflect the beliefs of financial participants about the risk of a joint default of financial institutions. Such beliefs incorporate forecasts of policy...
events as well as economic events, and therefore caution needs to be taken when using these bounds to orient the decision of policymakers.

At the same time, since the bounds presented in this paper can be constructed in real time, they are a useful tool that can be used to complement other measures in tracking the market’s perceptions of systemic risk. In addition, the possibility of tracking the whole structure of the financial network can be a valuable tool for identifying the sources of distress among banks at the core of the financial system.
References


Huang, Jing-zhi, and Ming Huang, 2003, “How much of the corporate-Treasury yield spread is due to credit risk?”, Working paper, Penn State University.


Appendix

Appendix A - Collateral agreements and the pricing of counterparty risk

In the text, I argue that the collateral agreements used for CDS contracts during the financial crisis were unlikely to eliminate counterparty risk. Buyers of CDS protection were aware of this and possibly priced it into the spreads. Here I report some evidence for the main points of the argument.

An initial question is whether counterparty risk was perceived at all by market participants. The growth of the percentage of OTC derivative contracts covered by some form of collateral confirms this indirectly: for credit derivatives, the volume-weighted percentage of collateralized contracts went from 39% in 2004 to 58% in 2005, to 66% in 2007 and 2008 (ISDA Margin Survey 2006, 2008). Besides, documents and interviews from practitioners directly confirm that the issue was taken into account by financial participants throughout the crisis. Robert McWilliam, head of Counterparty Risk management at ABN Amro, reports in January 2008:\(^1\) “The golden rule is to start early. If you start worrying about the counterparty when they are under duress your options are fairly limited”. A document from Barclays dated February 2008\(^2\) states:“While the maximum potential loss to the seller of protection is the contract spread for the rest of the contract duration, the buyer of protection could arguably lose the full notional of the contract (in case of simultaneous defaults by counterparty and the reference credit and zero recovery). Thus, counterparty risk is evidently more of a concern for buyers of protection.”

Even if agents were aware of counterparty risk, it was standard practice to ask for relatively little collateral, especially from the largest counterparties. ISDA reports that only about 2/3 of the contracts were covered by a collateral agreement, up to 2009. Besides, calculations by Singh and Aitken (2009) and Singh (2010) show that, even at the end of 2009, large financial institutions still carried large under-collateralized derivative liabilities. In particular, they compute the total value of “residual derivative payables” - liabilities from derivative positions after netting under master netting agreements and in excess of the collateral posted. For the 5 largest US dealers this amount was more than $250bn. Even though these numbers include all derivative contracts, and not only CDSs, they suggest a general under-collateralization of derivative positions from these counterparties. As an example of this, in 2008 Goldman Sachs had received collateral for 45% of the value of its receivable OTC derivatives, but posted only 18% of the value of payables. Similarly, JP Morgan in the same year had received collateral for 47% of receivables, but posted only 37% of the payables\(^3\). Finally, as reported in the main text, even the most active dealer in counterparty risk management, Goldman Sachs, failed to cover the full value of exposure on its CDS position with AIG.

Even when a collateral agreement is in place and actively managed, residual counterparty risk cannot be eliminated when the value of the derivative is subject to jumps. While during the crisis we did see gradual increases in CDS spreads of banks, a crucial episode - the

\(^{1}\)Reuters, “Banks move to guard against counterparty failures”, Jan 24, 2008.

\(^{2}\)Barclays Capital, 2008, “Counterparty Risk in Credit Markets”.

Lehman bankruptcy - shows that correlated jumps in credit risk (and defaults) are indeed possible. Just before the weekend of the 13th and 14th of September 2008, many institutions were considered at risk, but neither the credit ratings nor the CDS spreads indicated an extremely high likelihood of immediate default. For example, the Lehman 5 year CDS was trading at around 700bp per year, Merrill’s at 400bp, and the credit ratings of their debt were still as high as 4 months before, with an implied default probability of less than 0.25% per annum. A buyer who bought a Lehman or a Merrill CDS at 350bp per year a month before the default would have seen the value of the contract (the present discounted value of the difference in spreads) grow to 15 cents and 5 cents on the dollar respectively on Friday September 12th. Therefore, even if the buyers had called for enough collateral to cover the current value of such contracts, they would have improved their recovery rate by only 5% to 15%.

For the reasons explained above, buyers were generally aware that the collateral agreements in place (if any) would have left them exposed to the risk of double default. In fact, several sources document that in early 2008 buyers of CDS contracts were buying additional CDS contracts against their counterparties to hedge the residual counterparty risk. For example, from the documents on the AIG bailout (Maiden Lane III) from the Financial Crisis Inquiry Commission, we see that starting November 2007, Goldman Sachs - which had bought $22bn of CDS on a super-senior tranche of a CDO from AIG - was adjusting the amount of CDS protection against AIG together with their margin calls to AIG (which were caused by increases in the default probability of the underlying asset). Up to June 2008, the nominal amount of protection bought against AIG was of the same order of magnitude as the total amount of collateral called by Goldman.

In a document issued by Goldman Sachs in 2009 regarding the AIG bailout4, the firm declares: “In mid-September 2008, prior to the government’s action to save AIG, a majority of Goldman Sachs’ exposure [current market value] to AIG was collateralized and the rest was covered through various risk mitigants. Our total exposure on the securities on which we bought protection was roughly $10 billion. Against this, we held roughly $7.5 billion in collateral. The remainder was fully covered through hedges we purchased, primarily through CDS for which we received collateral from our market counterparties. Thus, if AIG had failed, we would have had the collateral from AIG and the proceeds from the CDS protection we purchased.”. Similarly, in an interview with ABN Amro, Reuters reports5: When counterparties [to OTC derivatives] are large corporations, which do not usually put up collateral, ABN buys protection in the CDS market against the default of the counterparty itself. ABN’s trading desk must go into the market constantly to rebalance those CDS holdings so that its protection equals its counterparty risk profile.”.

This evidence indicates that buyers were understanding the direct and indirect costs of the residual counterparty risk. Note that the fact that collateral was not enough to eliminate counterparty risk does not mean that buyers were making a bad deal on their contracts. Simply, they would have been compensated by paying a lower spread for the contracts when the counterparty was at higher risk of double default. In fact, the 2008 Barclays report titles a section: How much should I pay for a higher-rated counterparty? (The analysis then

---


quantifies this number for generic corporate reference entities of different credit rating).

Appendix B - Implementation of the Linear Programming Problem

This appendix describes in detail the algorithm employed to transform the probability bounds problem into a linear programming problem. It also describes the bond pricing formula and the linear approximation to the CDS pricing formula that allows to write the CDS constraints as linear constraints.

B.1 - Linear programming representation in the general case

This section describes the algorithm used to transform the probability problem

\[
\max P_r
\]

s.t.

\[
P(A_i) = a_i
\]

\[
\ldots
\]

\[
P(A_i \cap A_j) = a_{ij}
\]

into the LP representation

\[
\max p^T c_r
\]

s.t.

\[
p \geq 0
\]

\[
i^T p = 1
\]

\[
A p = b
\]

for the general case of N banks.

Start with a matrix \( B \) of size \((2^N, N)\) whose rows contain the binary representation of all numbers between 0 and \(2^N - 1\). For example, with \(N=4\):

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\vdots \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

Each row of this matrix corresponds to a particular element of the partition of the sample space described in Proposition 2: the event

\[
A_1^* \cap A_2^* \cap \ldots \cap A_N^*
\]

where \(A_j^* = A_j\) if element \(j\) of the row is 1, and \(A_j^* = A_j^c\) if element \(j\) of the row is 0.
The probability system \( p \) will then be determined as a vector of \( 2^N \) elements containing the probability of each of the elements of the partition represented by the \( 2^N \) rows of the matrix \( B \). For example \( p_1 \) will be the probability that none of the \( A_i \) events occur, \( p_2 \) will represent the probability that event \( A_N \) occurs but none of the other events does, and so on. Finally, the element \( p_{2^N} \) will represent the probability that all events occur.

The maximization problem presented above tries to find the vector \( p \) that maximizes the probability of systemic event of degree \( r \) \( (P_r) \) while satisfying constraints on marginal and pairwise default probabilities, as well as the constraints implied by the consistency of the probability measure. The latter are immediate: because the events represented by the rows of \( B \) are a partition of the sample space, and \( p \) is a probability measure on these events, all elements of \( p \) need to be nonnegative and sum to one:

\[
p \geq 0
\]

\[
p' = 1
\]

To obtain in LP form the inequalities and equalities that involve marginal and pairwise default probabilities, note first that because the elements of the partition are disjoint events, the probability of any union of them is equal to the sum of their probabilities. Therefore, to find the probability of an event \( A_i \), \( P(A_i) \), in terms of \( p \), one needs to sum the probabilities of all the elements of the partition in which event \( A_i \) occurs. But this is immediate given the representation in \( B \):

\[
P(A_i) = \sum_{j: B(j,i) = 1} p_j
\]

or:

\[
P(A_i) = a^t p
\]

for a vector \( a_i \) of size \((2^N, 1)\) s.t.:

\[
a_j^i = B(j,i)
\]

In other words, to find which elementary events form event \( A_i \) one needs to find all the rows of \( B \) in which element \( i \) is equal to 1. The union of these events will coincide with \( A_i \), and therefore the sum of their probabilities will be \( P(A_i) \). Given the linearity, this sum is equivalent to the product of the vector \( p \) with a vector \( a_i \), whose elements are ones whenever the corresponding elementary event is a subset of \( A_i \).

Similarly, the probability of a joint event:

\[
P(A_i \cap A_k) = \sum_{j: B(j,i) = 1 \text{ and } B(j,k) = 1} p_j
\]

or:

\[
P(A_i \cap A_k) = b^{ik} p
\]

for a vector \( b_{ik} \) of size \((2^N, 1)\) s.t.:

\[
b^i_j = B(j,i)B(j,k)
\]

i.e., the probability of the joint default is obtained summing the elements of \( p \) s.t. the
corresponding element of the partition involves both the occurrence of $A_j$ and of $A_k$. All these constraint can then be collected in the matrix form $Ap = b$.

Finally, the probability that at least $r$ events occur can be found as follows:

$$Pr = \sum_{j: (\sum_{h=1}^{N} B(j,h)) \geq r} p_j$$

or:

$$Pr = c^r p$$

for a vector $c^r$ of size $(2^N, 1)$ s.t.:

$$c^r_j = \left[ \sum_{h=1}^{N} B(j,h) \geq r \right]$$

where $[]$ is the indicator function.

Given this decomposition, the LP representation follows immediately.

**B.2 - Symmetry of the probability system**

Consider the vector $p \in \mathbb{R}^{2^N}$ representing a probability system on the $\sigma$-algebra generated by the basic events $A_1, ..., A_N$, as in Proposition 1. Consider a permutation $J$ of the indices of the basic events: $A_{J_1}, ..., A_{J_N}$, and call $\mathcal{M}$ the set of permutations. Call $p_J \in \mathbb{R}^{2^N}$ the vector representing the probability system generated by $A_{J_1}, ..., A_{J_N}$ that corresponds to $p$, constructed as in Proposition 1.

For example, take two events $A_1$ and $A_2$. The vector $p$ would have four elements: $p_1 = P(\overline{A_1} \cap \overline{A_2})$, $p_2 = P(\overline{A_1} \cap A_2)$, $p_3 = P(A_1 \cap \overline{A_2})$ and $p_4 = P(A_1 \cap A_2)$. In this case, only one additional permutation of the generating events is possible, $J = \{2, 1\}$, with $p_{J_1} = p_1$, $p_{J_2} = p_3$, $p_{J_3} = p_2$, and $p_{J_4} = p_4$.

**Definition.** A linear combination of the elements of $p$ defined by the vector $c$ is symmetric with respect to the generating events $A_1, ..., A_N$ if $c^T p = c^T p_J \forall J \in \mathcal{M}$. A linear programming problem, $\max c^T p \text{ s.t. } Ap \leq b$, is symmetric if $c$ and all rows of $A$ are symmetric with respect to the generating events $A_1, ..., A_N$.

An example of a symmetric weighting vector $c$ is the one corresponding to the probability of the union of the events, $c = [1 1 0 1]^T$, since $c^T p = c^T p_J = P(A_1 \cup A_2)$.

**Definition.** A probability system $p$ is symmetric if every event in $V$, the finest partition of the sample space generated by the basic events, has the same probability in all permutations of the generating events.

For example, with three generating events ($N = 3$), the probability system is symmetric if $P(A_1) = P(A_2) = P(A_3)$ and $P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_1 \cap A_3)$. I can now prove the following proposition:

**Proposition.** Suppose that the probability bounds correspond to a symmetric LP problem. Then, the bounds are attained by a symmetric probability system.
Proof. Start from a symmetric LP problem

\[
\max c^t p \\
\text{s.t.} \quad Ap \leq b
\]

Suppose that \(p^*\) is a solution to the problem. Given the definition of symmetry presented in the text, it is clear that \(p_J^*\) is also a solution to the problem: \(c^t p^* = c^t p_J^*\) and similarly hold for every row of the constraints, for every \(J\).

Now, construct \(p^{**}\) as follows:

\[
p^{**} = \frac{1}{\#J} \sum_J p_J^*
\]

where the first \(J\) correspond to no permutation, and \(J\) cycles across all permutations of indices \(A_1, ..., A_N\).

Note that it is also possible to construct \(p^{**}\) in the following way, considering the binary representation introduced in Proposition 2. Every \(b_i\) vector has \(O_i\) ones and \(N-O_i\) zeros. Call \(H_i\) the set of all vectors of size \(N\) that have \(O_i\) ones and \(N-O_i\) zeroes in different positions. Call \(b_{ih}\) the vector corresponding to element \(h\) from \(H_i\). Then, for every \(i\), construct \(p^{**}\) as:

\[
p_i^{**} = \left( \frac{O_i}{N} \right)^{-1} \sum_{h \in H_i} b_{ih}
\]

From the first construction, it is clear why \(p^{**}\) is a solution to the maximization problem, being just an average of solutions. Additionally, \(p^{**}\) is symmetric, which proves the statement of the Proposition.

An example with \(N = 3\). We can construct the probability system \(p^*\) as follows:

\[
p_1^* = Pr\{\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3\} \\
p_2^* = Pr\{\overline{A}_1 \cap \overline{A}_2 \cap A_3\} \\
p_3^* = Pr\{\overline{A}_1 \cap A_2 \cap \overline{A}_3\} \\
p_4^* = Pr\{A_1 \cap \overline{A}_2 \cap \overline{A}_3\} \\
p_5^* = Pr\{A_1 \cap \overline{A}_2 \cap A_3\} \\
p_6^* = Pr\{A_1 \cap A_2 \cap \overline{A}_3\} \\
p_7^* = Pr\{A_1 \cap A_2 \cap A_3\} \\
p_8^* = Pr\{A_1 \cap A_2 \cap A_3\}
\]

Suppose \(p^*\) solves the maximization problem, and construct \(p^{**}\) as:

\[
p_1^{**} = p_1^*
\]
\[
\begin{align*}
p_2^{**} &= p_3^{**} = p_5^{**} = \frac{p_2^* + p_3^* + p_5^*}{3} \\
p_4^{**} &= p_6^{**} = p_7^{**} = \frac{p_4^* + p_6^* + p_7^*}{3} \\
p_8^{**} &= p_8^*
\end{align*}
\]

\(p^{**}\) solves the maximization problem and is symmetric.

\[\square\]

**Corollary.** The bounds on systemic events of the type “at least \(r\) institutions default” given a symmetric constraint set (for example, constraints on the average marginal and pairwise default probabilities) are attained by a symmetric probability system.

The bounds obtained in a symmetric network in which we observe all marginal and pairwise probabilities will always be at least as wide as those obtained in an asymmetric network with the same averages of the low-order probabilities. The difference between the bounds obtained in the two cases captures precisely the extent to which asymmetry in the shape of the network affects the probability of systemic events.

**B.3 - Uniqueness of the solution and representative solution**

As described in section 3, it is possible to describe the range of possible values for a particular probability \(P(A') = v'p\) among the probability systems \(p\) that attain the solution to the probability bounds problem:

\[
c_{\text{max}} = \max_p c_r'p
\]

s.t.

\[
p \geq 0 \\
v'p = 1 \\
Ap = b
\]

(and similarly for the lower bound). To find the maximum (minimum) value of \(v'p\) at the upper bound on systemic risk, simply solve the max (or min) problem

\[
\max_p (\min_p v'p)
\]

s.t.

\[
p \geq 0 \\
v'p = 1 \\
Ap = b
\]

7
To find the maximum and minimum value of $v'p$ at the lower bound, replace the last constraint with $c_r'p = c_{\min}$.

Appa (2002) describes an algorithm (called AFROS) to find a “representative” solution to the bounds. Call $Ep = d$ the set of all equality constraints in the problem, and note that the original maximization problem can always be rewritten with a series of equality constraints and a nonnegativity constraint for an expanded vector $p$. The algorithm proceeds as follows.

1. Solve $P^0$: \[ \max c'_r p \text{ s.t. } Ep = d, p \geq 0. \] Let $p = p^0$ be the solution to $P^0$.

2. Solve $P^1$: \[ \max d^1 p \text{ s.t. } Ep = d, p \geq 0, c'_r p = c'_r p^0, \] if $p^0_j = 0$, otherwise $d^1_j = 0$. If maximum value of $d^1 p = 0$, stop. There is no alternative solution. Otherwise, let $p = p^1$ be the optimal solution to $P^1$. Set counter $q = 1$.

3. $q = q + 1$. While $q < s$, solve $P^q$: \[ \max d^q p \text{ s.t. } Ep = d, p \geq 0, c'_r p = c'_r p^0, d^r p \geq \alpha^r \text{ for } r = 1, ..., q - 1. d^q_j = 1 \text{ if } p^{q-1}_j = 0, 0 \text{ otherwise, and } \alpha^r \text{ are positive numbers.} \]

4. After $S$ steps, the “representative solution” is the average of the $p$ from $p^0$ to $p^S$.

The algorithm starts from a solution $p^0$ and looks for another solution that maximizes the sum of all and only those elements of $p$ that are hitting the constraint $p \geq 0$ at $p^0$. This is the sense in which $p^1$ will be as dissimilar as possible from $p^0$. From $p^1$, it will repeat the same procedure to find another solution $p^2$, and so on, with additional constraints that ensure that the algorithm never converges back to a previous solution. The average of these solutions, itself a probability system, is the “representative” solution.

Appendix C: additional pricing details

C.1 Bond pricing

In this section I show how to obtain the bond pricing formula in the text starting from the general discrete-time formulation of bond pricing in the reduced form model. Call $h_t^i$ the hazard rate process for $i$: the probability of default in month $t$ conditional on survival until then. Call $r_{t,T}^F$ the monthly risk-free process. Call the $r_{t,T}^F$ the realized return of the short-term risk-free security between $t$ and $T$, s.t.

\[
(1 + r_{t,T}^F) = \prod_{s=t+1}^{T} (1 + r_s^F)
\]
The time t price of a risk-free zero-coupon bond of face value $1 at time T is:

\[ \delta(t, T) = E^Q_t \left[ \frac{1}{1 + r_{t,T}} \right] \]

Call \( G^i(t, s) \) the probability of survival of firm i up to s under a certain realization of hazard rates of default, i.e.:

\[ G^i(t, s) = \prod_{r=t}^{s-1} (1 - h^i_r) \]

where \( E^Q \) indicates the expectation taken under the risk neutral probability measure.

The price of a liquid bond \( j \) issued by firm i of face value $1, maturity \( T_{ij} \), coupon rate \( c_{ij} \) and recovery equal to a fraction \( R_{ij} \) of the value of a Treasury zero-coupon bond of comparable maturity is:

\[
B^{ij}(t, T_{ij}) = E^Q_t \left[ \sum_{s=t+1}^{T_{ij}} \frac{G^i(t, s) c^j_{ij}}{1 + r_{t,s}^F} + \sum_{s=t+1}^{T_{ij}} \frac{G^i(t, s-1) h^i_{s-1} R_{ij}}{1 + r_{t,s}^F} \right]
\]

In this paper I consider only senior unsecured bonds of different coupons and maturities, so I assume that the recovery rate is the same for all bonds and that it is also the same for similar bonds of other firms in the financial industry, i.e. \( R_{ij} = R \).

We can then add a liquidity process \( \gamma^i_t \), assumed to be the same for all bonds of equal seniority of firm i. Following Duffie (1999), this liquidity cost will appear in the bond pricing equation as a per-period proportional cost incurred while holding the bond.

In theory, it is possible to write down a parametric version of the (generally not independent) processes that govern the evolution of \( r^F_t, h_t, \gamma_t \), and additionally a time-varying recovery rate. Examples of this can be found in Duffie and Singleton (1997) and Longstaff, Mithal and Neis (2005). In this paper, I use a simplified pricing model that assumes that, for any given firm, the prices of all of its bonds are determined independently at each time t, under the assumption that from time t onwards \( h_{t+s} \) and \( \gamma_{t+s} \) will be constant and equal to \( h_t \) and \( \gamma_t \), respectively. This is just an approximation, because prices do not take into account that at each future date these parameters are going to be revised, since at every future date \( t + r \) prices will be recomputed assuming a constant hazard rate and liquidity process from \( t + r \) on at new levels, \( h_{t+r} \) and \( \gamma_{t+r} \). I discretize the model to a monthly horizon, and I assume that coupons are paid monthly. Further assuming independence of the risk-free rate process from all other processes (under Q) we obtain:

\[
B^{ij}(t, T_{ij}) = c^j_{ij} \left( \sum_{s=t+1}^{T_{ij}} \delta(t, s)(1 - h^i_t)^{s-t}(1 - \gamma^i_t)^{s-t} \right) + \\
+ \delta(t, T_{ij})(1 - h^i_t)^{T_{ij}-t}(1 - \gamma^i_t)^{T_{ij}-t} + R \left( \sum_{s=t+1}^{T_{ij}} \delta(t, s)(1 - h^i_t)^{s-t-1}(1 - \gamma^i_t)^{s-t-1} h^i_t \right)
\]
C.2 - Additional details of CDS contracts

Besides those considered explicitly in this paper, there are other elements of CDS contracts that potentially affect their spreads. First, liquidity of the CDS market could influence the CDS spreads, just as bond liquidity is known to affect bond prices. In this paper, I explicitly take into account liquidity premia in bond prices, but not in CDS spreads. For the case of CDSs, liquidity is much less likely to be an issue, especially because they require much less capital at origination and they are not in fixed supply.\footnote{For an additional discussion of this and on the supporting evidence, see Blanco, Brennan and Marsh (2003,2005).}

Also, I abstract from restructuring clauses and the cheapest-to-deliver option sometimes present in CDS contracts. A restructuring clause (under which payment is triggered for simple debt restructuring, in addition to bankruptcy) is more frequent for European bonds, and this results in the contract being triggered in cases close to the Chapter 11 for the US. Berndt, Jarrow and Kang (2007) estimate that the presence of such clause increases the value of the CDSs by 6-8%, and all the results in this paper are robust to an adjustment of CDS spreads of that magnitude. The value of the cheapest-to-deliver option (which allows the buyer to deliver to the seller the cheapest of the defaulted bonds of the same seniority as the reference bond) will be small relative to the CDS spread as long as in default all senior unsecured bonds have similar recovery rates. Additionally, as observed in the Delphi and Calpine defaults in 2005, the high demand for the cheapest bonds might determine shortages of such securities and therefore, anticipating this, a reduction in the ex-ante value of the option.\footnote{De Wit (2006).}

C.3 - CDS pricing approximation

Start from the discretized pricing equation with constant hazard and risk-free rates, starting from period 0 for notational simplicity.

\[
\sum_{s=1}^{T} \delta(0, s - 1)(1 - P(A_i \cup A_j))^{s-1} z_{ji} =
\]

\[
= \left[ \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))\delta(0, s)(1 - R) \right]
\]

We can rewrite the equation as:

\[
\frac{z_{ij}}{(1 - R)} = \frac{\sum_{s=1}^{T} \delta(0, s)(1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))}{\sum_{s=1}^{T} \delta(0, s - 1)(1 - P(A_i \cup A_j))^{s-1}}
\]

and then approximate the right hand side around \( P(A_i) = 0, P(A_j) = 0, P(A_i \cap A_j) = 0, \)
remembering that
\[ P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i \cap A_j) \]

To obtain the approximation, note that while we cannot vary \( P(A_i) \) and \( P(A_i \cap A_j) \) independently around 0, we can rewrite the expression in terms of:

\[ \pi_i = P(A_i \cap A_j) \]
\[ \pi_j = P(\overline{A_i} \cap A_j) \]
\[ \pi_{ij} = P(A_i \cap A_j) \]

which can be varied independently of each other. The right-hand side (call it \( G \), and call \( G(0) \) the function at the approximation point) can then be written as:

\[
G = \frac{\sum_{s=1}^{T} \delta(0, s)(1 - \pi_i - \pi_j - \pi_{ij})^{s-1}(\pi_i + \pi_{ij} - (1 - S)\pi_{ij})}{\sum_{s=1}^{T} \delta(0, s - 1)(1 - \pi_i - \pi_j - \pi_{ij})^{s-1}}
\]

First, note that \( G(0) = 0 \). Second, take \( G_{\pi_i}(0) \). It is easy to show that

\[
G_{\pi_i}(0) = \frac{\sum_{s=1}^{T} \delta(0, s)}{\sum_{s=1}^{T} \delta(0, s - 1)}
\]

Similarly:

\[
G_{\pi_j}(0) = \frac{\sum_{s=1}^{T} \delta(0, s) \frac{d}{d\pi_j} [(1 - \pi_i - \pi_j - \pi_{ij})^{s-1}(\pi_i + \pi_{ij} - (1 - S)\pi_{ij})]}{\left[\sum_{s=1}^{T} \delta(0, s - 1)\right]^2}
\]

with

\[
\frac{d}{d\pi_j} [(1 - \pi_i - \pi_j - \pi_{ij})^{s-1}(\pi_i + \pi_{ij} - (1 - S)\pi_{ij})] = \left[ \frac{d}{d\pi_j} (1 - \pi_i - \pi_j - \pi_{ij})^{s-1} \right] [(\pi_i + \pi_{ij} - (1 - S)\pi_{ij})] = 0
\]

so that

\[ G_{\pi_j}(0) = 0 \]

Finally we get:

\[
G_{\pi_{ij}}(0) = \frac{\sum_{s=1}^{T} \delta(0, s) - S}{\sum_{s=1}^{T} \delta(0, s - 1)}
\]

So that:

\[
G \simeq \frac{\sum_{s=1}^{T} \delta(0, s)}{\sum_{s=1}^{T} \delta(0, s - 1)} [\pi_i + S\pi_{ij}] = \frac{\sum_{s=1}^{T} \delta(0, s)}{\sum_{s=1}^{T} \delta(0, s - 1)} [\pi_i + \pi_{ij} - (1 - S)\pi_{ij}]
\]

\[
= \frac{\sum_{s=1}^{T} \delta(0, s)}{\sum_{s=1}^{T} \delta(0, s - 1)} \left[ P(A_i) - (1 - S)P(A_i \cap A_j) \right]
\]
The result is:

\[
\frac{z_{ij}}{(1 - R)} \approx \left[ \frac{\sum_{s=1}^T \delta(0, s)}{\sum_{s=1}^T \delta(0, s - 1)} \right] (P(A_j) - (1 - S)P(A_i \cap A_j))
\]

It is important to check the accuracy of the approximation for a realistic range of parameters. For several different points in time (every 50 days) between 1/1/2007 and 3/31/2009, I compare the correct spread and the approximated spread, computed using the US yield curve at that time, considering:

- different values of \(P(A_j)\): between 0 and the maximum probability implied by bond data under no liquidity assumptions (\(\max_j \{h_j(0)\}\)).
- different values of \(P(A_i \cap A_j)\): between 0 and \(P(A_j)\)
- different values of \(R\) and \(S\): between 0.1 and 0.4.

In all these simulations, the approximation error is between 0.2% and 0.3% of the true value of the CDS spread.

**Appendix D - Robustness Tests**

In this section I study the robustness of the main results of the paper to different assumptions. In Appendix Table 1, I report the average value of the bounds for different subperiods, chosen to reflect the main events identified in the Figures. For each robustness test (all performed under the calibration of the liquidity to the basis of nonfinancial institutions, \(\gamma^i_t = \alpha_t^i \lambda^i_t\)) I report in Appendix Table 1 the average value of the bounds, in basis points per month, during different periods: January to December 2007, January 2008 to March 15 2008 (the run-up to Bear Stearns’ collapse), from Bear’s episode to Lehman’s default (on September 15th 2008), the month after Lehman’s default (in which CDS spreads and bond yields spiked), the period between September 2008 and April 2009 (the latest peak of the crisis, just before the stress test results were released) and finally from May 2009 to June 2010. In the three panels, I show values for the lower and upper bound on \(P_1\), and the upper bound on \(P_4\) (the lower bound on \(P_4\) is always 0).

Besides showing the level and the time series of the bounds, this Table allows us to check that the main results reported in the paper hold under different assumptions. The bold line in each panel of Appendix Table 1 reports the baseline case presented in the paper. We can confirm the result, presented in Figure 5 (third panel), that systemic risk was low in the months preceding Bear Stearns’ collapse, while idiosyncratic risk was already high. Besides, we can see that during the month after Lehman’s default, idiosyncratic risk spiked (it increased sharply and then decreased as sharply). The upper bound on \(P_4\) increases as
well, but does not even reach the level observed in the following 6 months: systemic risk keeps increasing until March 2009.

D.1 - Assumptions on S

Let us start with robustness with respect to the assumed recovery rate of CDSs when double default occurs, \( S \in [R, 1] \). The effect of changes in this assumption depends crucially on the liquidity-adjusted bond/CDS basis of each bank. For some banks, the basis is small enough that it can be completely explained by counterparty risk. For these banks, an increase in \( S \) means that the same basis can account for higher counterparty risk. For other banks, instead, the basis is large enough that, due to internal constraints of the probability system, it cannot be completely explained by counterparty risk: even at the upper bound for systemic risk, a part of the basis must be explained by liquidity. For these banks, an increase in \( S \) means that the same amount of counterparty risk - which was already at the maximum possible - will explain an even smaller fraction of the basis. This means that the marginal probability of default, \( P(A_i) \), has to decrease. In turn, this directly reduces the maximum possible amount of counterparty risk for contracts written by \( i \) against other banks, since for each \( j \) we must have \( P(A_i \cap A_j) \leq P(A_i) \).

An increase in \( S \) then has a different effect on banks with a relatively small basis and banks with a large basis. The two effects are also at play for each bank individually, for different starting levels of \( S \): when \( S \) is low enough counterparty risk has a large effect on CDS spreads, and therefore the basis will be relatively small - it can be completely explained by counterparty risk. When \( S \) is large enough, not all the basis can be explained by counterparty risk, and the second mechanism operates. Typically, because of asymmetry in the basis across banks, for most values of \( S \) the two effects described above will operate for some banks in one direction and for other banks in the opposite direction. This explains why we see the bounds on systemic risk being very robust to changes in \( S \) (at least up to a recovery rate of 90%), as shown in Appendix Table 1.

To see formally the effect of \( S \) on the implied estimate of systemic risk, it is useful to look at a symmetric network. Remember that the upper bound on systemic risk is attained by the most correlated probability system that satisfies the constraints:

\[
P(A_i) \leq h_i(\gamma_i^*)
\]

\[
P(A_i) - (1 - S) \left( \frac{1}{N-1} \sum_{j \neq i} P(A_i \cap A_j) \right) = b_i
\]

where \( b_i = \frac{\sum_{s=1}^{T} s \delta(0,s-1)}{\sum_{s=1}^{T} s \delta(0,s)(1-R)} \).

\[8\] I focus on the upper bound for the probability of at least \( r > 1 \) events occurring. Following the analysis reported in section 3, the same argument holds for the lower bound for the probability that at least 1 institution defaults, since that is achieved for a very correlated system. It is easy to see why the results for the lower bound for \( r > 1 \) and the upper bound for \( r = 1 \) do not depend on \( S \): these bounds look for the least correlated system, which can always be obtained by setting the marginal default probabilities at the levels implied by the
Intuitively, for a given $S$, one can obtain the most correlated probability system by setting $P(A_i)$ as high as possible (up to the constraint $h_i(\gamma_i))$ for all banks and then increasing the term $\frac{1}{N-1}\sum_{i \neq j} P(A_i \cap A_j)$ to match the CDS spreads $(b_i)$. Counterparty risk would explain the whole bond/CDS basis, and a higher recovery rate $S$ would imply that a higher joint default probability is needed to match it, increasing the upper bound on systemic risk. This intuitive reasoning, however, does not take into account the internal restrictions of consistency of the probability system. For a symmetric network, call the marginal probability of default of each bank $q_1$ and the pairwise joint probabilities of default of each pair $q_2$. The previous constraints become:

$$q_1 \leq h$$
$$q_1 - (1 - S)q_2 = b$$

where $h$ is the (common) upper bound on the marginal probability of default and $b$ is the (common) $b_i$.

To maximize systemic risk, we would intuitively set $q_1 = h$, and then $q_2$ will be set to match CDS spreads:

$$q_2 = \frac{q_1 - b}{1 - S}$$

For given $q_1$, $q_2$ is increasing in $S$, as is systemic risk. This captures the intuition that a higher recovery rate of CDSs implies that higher counterparty risk is needed to explain the same bond/CDS basis.

In fact, this effect is at play only when $S$ is small enough. As $S$ grows, $q_2$ keeps increasing, and at some point it will reach the level $q_2 = q_1$. At that point, the internal consistency of the probability system kicks in, preventing further increases: it would violate the implicit constraint that $q_2 \leq q_1$.

What happens then if $S$ increases further? The only way to satisfy the constraints is to lower $q_1$ below $h$: for $q_1 = h$ there might exist no probability systems able to satisfy both constraints: matching the CDS spread and satisfying internal consistency. Instead, with a lower $q_1$, it is possible to set $q_2$ to be equal to $q_1$ and satisfy the CDS constraint, so that:

$$q_2 = q_1 = \frac{b}{S}$$

which is decreasing in $S$. This means that for large enough values of $S$, the bond/CDS basis is too large to be explained by counterparty risk. Even at the upper bound on systemic risk, liquidity has to explain part of the basis. In a symmetric system, then, the bounds on systemic risk first increase and then decrease with $S$.

These forces play out in similar but nonlinear ways for asymmetric networks. In that case, the asymmetry in the bond/CDS basis across banks means that the upper bounds on marginal probabilities (that are obtained from bond prices) will bind for some banks and not for others, so that the overall effect of a change in $S$ on the measure of systemic risk will be small.

CDS spreads and attributing the bond/CDS basis entirely to liquidity.
D.2 - Assumptions on R

The case for the recovery rate of bonds $R$ is different. As shown in Section C of the Appendix, $R$ affects the prices of both bonds and CDSs. A higher expected recovery rate in case of default increases the value of a bond, and at the same time decreases the value of CDS insurance written on that bond, since the payment from the CDS seller covers only the amount of bond value not recovered in default. Because this recovery rate multiplies the marginal and joint default probabilities in the pricing formulas, when $R$ changes all probabilities implied in bonds and CDSs are scaled up or down by approximately the same amount.\(^9\) Therefore, the bounds on systemic risk will scale in a similar way. However, the main results on the time series of the bounds will not change, as shown by Appendix Table 1.

D.3 - Time varying recovery rates

Above I have studied robustness to different assumptions about $S$ and $R$, when these are assumed to be constant during the whole sample period. In theory, it is possible that these recovery rates vary over time in a way that affects the results on the time-series of systemic risk presented in section 4. Suppose that at every time $t$ bonds and CDSs are still priced assuming that at all future periods $t+s$ the recovery rates are constant and equal to $S_t$ and $R_t$; however, let now $S_t$ and $R_t$ vary over time. How will this affect the bounds on systemic risk?

The tests presented above show that the bounds on systemic risk scale in the same direction as $R$. If we believe that, during peak episodes like the one following Lehman’s default, recovery rates $R$ might have dropped, this would in fact strengthen the result that the spike in systemic risk was then relatively low, because it would further reduce the bound on $P_4$ during that month.

Another possibility is a reduction in the recovery rate of CDSs, $S$, in times when systemic risk increases. However, it is easy to see that this case actually reinforces the main empirical results. If the recovery rate $S$ becomes smaller during the key episodes of the crisis, then joint default risk has to be smaller as well. This stems once more from the fact that during these episodes the bond/CDS basis is small relative to CDS and yield spreads. When $S$ is reduced, the probability of joint default has a greater effect on the basis. To still match the basis even if $S$ is higher, joint default risk has to decrease. Therefore, the main results in the paper will be robust to a decrease in the recovery rate $S$ in times of crisis.

D.4 - Stochastic recovery rate on bonds $R$

Another possibility is that when pricing bonds and CDSs, agents incorporate the possibility that recovery rates might be stochastic and correlated with the default events in the financial sector. In particular, one could think that recovery rates of both bonds and CDSs

\(^9\)The difference between the two comes from differences in the cash flow timing of bonds and CDSs. They are scaled by exactly the same amount in the simple two-period example of section 2.
might deteriorate the more defaults happen in the financial system.

Because of the limited data available, it is difficult to solve explicitly for the case of stochastic recovery rates. However, it is possible to gain some intuition on the effect of this assumption under simple modeling assumptions. Suppose that the recovery rate on bonds is \( R_H \) whenever one bank defaults alone, and \( R_L < R_H \) whenever two or more banks default. Below I show that we can decompose as follows the change in the bounds on systemic risk, going from a non-stochastic recovery rate \( R \) to the stochastic recovery process described above. First, we can shift the (constant) recovery rate downwards for both bonds and CDSs to \( R_L \). This component scales down the bond-implied and the CDS-implied probabilities by a similar amount, as discussed above. This would scale the bounds on systemic risk downwards. Second, we increase the present value of bonds by an amount \( Y_{\text{bond}} \), and decrease the present value of payments of the CDS contract by an amount \( Y_{\text{CDS}} \approx Y_{\text{bond}} \) (in a first-order approximation with small probabilities of default). This second effect shifts the CDS spread and the yield spread in the same direction by a similar amount, with minimal effect on the basis and hence on counterparty risk. We then expect the bounds on systemic risk to become lower if we introduce a stochastic recovery rate with \( R_L < R \). The reason is that for the purpose of systemic risk, the relevant recovery rate is the one that obtains in states of multiple defaults, or \( R_L \) in this case. However, as long as the recovery rates \( R_L \) and \( R_H \) themselves do not vary over time, the time series of the bounds should still look as in Figures 4 and 5.

To obtain this decomposition, start by defining \( X_i \equiv \bigcup_{k \neq i} A_k \) the event of at least one default among the banks different from \( i \), and similarly \( X_{ij} \equiv \bigcup_{k \neq i,j} A_k \). Call \( B_R(0,T) \) the price of a bond under the assumption of constant recovery rate \( R \) and \( B_{R_L,R_H}(0,T) \) the price of a bond with stochastic recovery rate described above, and similarly for the CDS spreads. Then, it is easy to see that (setting liquidity to 0 for simplicity)

\[
B_{R_L,R_H}(0,T) = c \left( \sum_{t=1}^{T} \delta(0,t)(1 - P(A_i))^t \right) + \delta(0,T)(1 - P(A_i))^T + 
\]

\[
+ R_H \left( \sum_{t=1}^{T} \delta(0,t)(1 - P(A_i))^{t-1} P(A_i \cap \overline{X}_i) \right) + R_L \left( \sum_{t=1}^{T} \delta(0,t)(1 - P(A_i))^{t-1} P(A_i \cap X_i) \right)
\]

while the CDS spread solves:

\[
\sum_{s=1}^{T} \delta(0, s - 1)(1 - P(A_i \cup A_j))^{s-1} z_{R_L,R_H,j} = 
\]

\[
= \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s) [P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(1 - R_H) 
\]

\[
+ P(A_i \cap \overline{A}_j \cap X_{ij})(1 - R_L) + P(A_i \cap A_j)(1 - R_L)S]
\]

Now, rewrite bond prices as:
\[ B_{RL,RH}(0, T) = c \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^t \right) + \delta(0, T)(1 - P(A_i))^T + \\
+ (R_H - R_L) \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^{t-1} P(A_i \cap X_i) \right) + \\
+ R_L \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^{t-1} \left[ P(A_i \cap X_i) + P(A_i \cap X_i) \right] \right) \]

Noting that the term \( P(A_i \cap X_i) + P(A_i \cap X_i) = P(A_i) \), we can rewrite

\[ B_{RL,RH}(0, T) = c \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^t \right) + \\
+ \delta(0, T)(1 - P(A_i))^T + R_L \left( \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^{t-1} P(A_i) \right) + Y_{bond} \]

or:

\[ B_{RL,RH}(0, T) = B_{RL}(0, T) + Y_{bond} \]

where

\[ Y_{bond} = \sum_{t=1}^{T} \delta(0, t)(1 - P(A_i))^{t-1} P(A_i \cap X_i)(R_H - R_L) \]

The price of the bond is now equal to the price of the bond in case that the recovery rate is constant and equal to \( R_L \) plus the last term, which is the present value of the additional recoveries in case \( i \) defaults alone.

A similar formula holds for CDS spreads. Since \( P(A_i \cap \overline{A}_j \cap \overline{X}_{ij}) = P(A_i \cap \overline{X}_i) \), we can rewrite the CDS spread as

\[ \sum_{s=1}^{T} \delta(0, s - 1)(1 - P(A_i \cup A_j))^{s-1} z_{RL, RH, ji} = \]

\[ = \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s)[P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(1 - R_H) - P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(1 - R_L) \\
+ P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(1 - R_L) + P(A_i \cap \overline{A}_j \cap X_{ij})(1 - R_L) + P(A_i \cap A_j)(1 - R_L)S] \\
= \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s)[P(A_i \cap \overline{A}_j \cap \overline{X}_{ij})(R_L - R_H) \\
+ (1 - R_L) \left( P(A_i \cap \overline{A}_j \cap \overline{X}_{ij}) + P(A_i \cap \overline{A}_j \cap X_{ij}) + P(A_i \cap A_j)S \right)] \]

Since

\[ P(A_i \cap \overline{A}_j \cap \overline{X}_{ij}) + P(A_i \cap \overline{A}_j \cap X_{ij}) + P(A_i \cap A_j)S = \\
= P(A_i \cap \overline{A}_j) + P(A_i \cap A_j)S = P(A_i) - (1 - S)P(A_i \cap A_j) \]
we can write:
\[
\sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1} z_{RL, RH} = \\
= \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s)[P(A_i \cap \overline{A_j} \cap \overline{X_{ij}})(R_L - R_H) \\
+ (1 - R_L)(P(A_i) - (1 - S)P(A_i \cap A_j))] \\
= \left\{ \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s) (P(A_i) - (1 - S)P(A_i \cap A_j))(1 - R_L) \right\} - Y_{CDS}
\]
or:
\[
\sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1} z_{RL, RH} = \sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1} z_{RL, RH} - Y_{CDS}
\]
where
\[
Y_{CDS} = \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1} \delta(0, s) P(A_i \cap \overline{X_i})(R_H - R_L)
\]

Now we can show that when probabilities are small, \(Y_{CDS}\) and \(Y_{bond}\) are approximately the same. To do this, rewrite the two formulas in terms of the following probabilities:
\[
\pi_i = P(A_i \cap \overline{A_j} \cap \overline{X_{ij}}) = P(A_i \cap \overline{X_i}), \pi_j = P(A_j \cap \overline{A_i}), \pi_{ij} = P(A_i \cap A_j), \pi_k = P(A_i \cap \overline{A_j} \cap X_{ij}).
\]
Note that these three sets are disjoint and therefore if the probabilities of the other events are small we can vary them independently of each other. Then,
\[
Y_{CDS} = \sum_{s=1}^{T} (1 - \pi_i - \pi_j - \pi_{ij} - \pi_k)^{s-1} \delta(0, s) \pi_i (R_H - R_L)
\]
\[
Y_{bond} = \sum_{s=1}^{T} (1 - \pi_i - \pi_{ij} - \pi_k)^{s-1} \delta(0, s) \pi_i (R_H - R_L)
\]

Approximating each of these equations around \(\pi_i = \pi_j = \pi_{ij} = \pi_k = 0\), we note that:
\[
Y_{CDS}(0) = 0
\]
\[
\frac{d}{d\pi_i} Y_{CDS}(0) = \sum_{s=1}^{T} \delta(0, s)(R_H - R_L) \\
\cdot \left[ \left( \frac{d}{d\pi_i} (1 - \pi_i - \pi_j - \pi_{ij} - \pi_k)^{s-1} \right) \pi_i + ((1 - \pi_i - \pi_j - \pi_{ij} - \pi_k)^{s-1}) \frac{d}{d\pi_i} \pi_i \right]_0 \\
= \sum_{s=1}^{T} \delta(0, s)(R_H - R_L)
\]

18
\[ \frac{d}{d\pi_j} Y_{CDS}(0) = \frac{d}{d\pi_j} Y_{CDS}(0) = \frac{d}{d\pi_k} Y_{CDS}(0) = 0 \]

And similarly for \( Y_{bond} \): so, to a first approximation, \( Y_{CDS} = Y_{bond} \).

**D.5 - Assumptions about the hazard rate**

In this section I allow for a more flexible form for the hazard rate process. In particular, for each institution \( i \), from the perspective of an agent pricing bonds at time \( t \), the hazard rate at time \( t + s \) follows the (deterministic but time-varying) process

\[ h_{t+s} = (1 - \rho_t) \bar{h}_t + \rho_t h_{t+s-1} \] (2)

where parameters \( h_t, \bar{h}_t \) and \( \rho_t \) are determined at time \( t \). As before, all bonds are priced at every time \( t \) assuming that the hazard rate process is known for all future dates.

At each time \( t \), I can use bond prices to estimate \( h_t, \rho_t \) and \( \bar{h}_t \). This representation allows to capture cases in which the hazard rate is higher at shorter horizons and then reverts to a lower long-term value. I can therefore construct the bounds on systemic risk using \( h_t \), the probability of default in the month after \( t \).

As discussed in section 4, the main problem with this approach is that while it is easy to estimate a more flexible function for the marginal hazard rate of default using bond prices, CDS data do not contain enough information to estimate a similarly flexible process for joint default risk (because at each time \( t \) we only observe the spread of one CDS, with maturity of 5 years). To tackle this limitation, I assume that the joint hazard process replicates the shape of the marginal hazard process of the reference entity: the process decays at the same rate \( \rho_t \) and displays the same ratio between short-term and long-term default hazards \( (h_t/\bar{h}_t) \). Appendix Table 1 shows that the main empirical results are confirmed under these assumptions.

The derivation of the constraints of the maximization problem under these assumptions proceeds as follows. Start by noting that

\[ h_{t+s} = (1 - \rho_t)(1 + \rho_t^{s-1})\bar{h}_t + \rho_t^{s} h_t \]

\[ = (1 - \rho_t) \frac{1 - \rho_t^s}{1 - \rho_t} \bar{h}_t + \rho_t^s h_t \]

\[ = (1 - \rho_t^s) \bar{h}_t + \rho_t^s h_t \]

for \( s \geq t \).

The probability of surviving until \( t + r \) is

\[ H_t(t + r; h_t, \rho_t) = (1 - h_t)...(1 - h_{t+r}) \]

From the cross section of outstanding bonds, we can then estimate at each \( t \) the three parameters \( h_t, \rho_t, \bar{h}_t \).

Since CDS spreads depend on the process of joint hazard rate of default, but we only
observe the price of the 5-year CDS, I assume that the shape of the joint default hazard rate $h_{ij}^s$ is similar to that of the marginal hazard rate of the reference entity $i$ (similar results hold if we assume that it inherits the shape estimated from the bond prices of the seller, or a combination of both). In particular, after having estimated the three parameters for the hazard rate of bank $i$, I define

$$\alpha_i^t = \frac{\overline{h}_i^t}{h_i^t}$$

so that:

$$h_{t+s}^i = (1 - \rho_i^s T_i^t + \rho_i^s h_t^i) = (1 - \rho_i^s)\alpha_i^t h_t + \rho_i^s h_t = (\alpha_i^t - \rho_i^s \alpha_i^t + \rho_i^s) h_t$$

Call $h_i^t$ the probability that $i$ defaults, similarly for $j$ and finally $h_{ij}^s$ the probability of joint default. Assume that:

$$h_{t+s}^i = (1 - h_i^t + h_{ij}^t - h_i^s)\alpha_i^t + h_{ij}^t$$

$$h_{t+s}^j = (1 - h_j^t - h_{ij}^t - h_j^s)\alpha_i^t + h_{ij}^t$$

$$h_{t+s}^{ij} = (1 - h_{ij}^t - h_{ij}^s)\alpha_i^t + h_{ij}^t$$

Note that this requires $h_i^t > 0$ for $\alpha_i^t$ to be defined. Therefore, I impose a lower bound on $h_i^t$ of $10^{-6}$, or a hundredth of a basis point.

Define

$$H_{ij}^t(t + s; h_t, \rho_t) = (1 - h_i^t - h_j^t + h_{ij}^t)(1 - h_i^t - h_j^t + h_{ij}^t)...(1 - h_i^{t+s} - h_j^{t+s} + h_{ij}^{t+s})$$

which is the probability of having no credit events until time $t + s$, and

$$H_{ij}^t(t; h_t, \rho_t) = 1$$

The CDS spread at time $t$ satisfies:

$$z_{ij}^t = \left[ \frac{\sum_{s=1}^{T} \delta(t, t + s) H_{ij}^t(t + s - 1; \rho_i, h_t)(h_i^{t+s} - (1 - S)h_{ij}^{t+s})}{\sum_{s=1}^{T} \delta(t, t + s - 1) H_{ij}^t(t + s - 1; \rho_i, h_t)} \right]$$

We now approximate this formula around $h_i^t = h_j^t = h_{ij}^t = 0$. To do this, we rewrite the formula in terms of probabilities of disjoint events:

$$\pi_i = h_i^t - h_{ij}^t$$

$$\pi_j = h_j^t - h_{ij}^t$$

$$\pi_{ij} = h_{ij}^t$$

So we have:

$$h_{t+s}^i = (\alpha_i^t - \rho_i^s \alpha_i^t + \rho_i^s)(\pi_i + \pi_{ij})$$
\[ h_{t+s}^i = (\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s)(\pi_i + \pi_{ij}) \]

\[ h_{t+s}^{ij} = (\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)\pi_{ij} \]

Call

\[ G = \frac{\left[ \sum_{s=1}^{T} \delta(t, t + s) H_t^{ij}(t + s - 1; \rho_t, h_t)(h_{t+s}^i - (1 - S)h_{t+s}^{ij}) \right]}{\sum_{s=1}^{T} \delta(t, t + s - 1) H_t^{ij}(t + s - 1; \rho_t, h_t)} \]

We can start by noting that at the approximation point, \( \pi_i = \pi_j = \pi_{ij} = 0 \), we have:

\( (h_{t+s}^i - (1 - S)h_{t+s}^{ij})|_0 = 0 \)

\( H_t^{ij}(t + s - 1; \rho_t, h_t)|_0 = 1 \)

We then have, for each of the \( \pi_i \)

\[ \frac{d}{d\pi} G|_0 = \frac{\left[ \sum_{s=1}^{T} \delta(t, t + s) H_t^{ij}(t + s - 1; \rho_t, h_t)(h_{t+s}^i - (1 - S)h_{t+s}^{ij}) \right]}{\left[ \sum_{s=1}^{T} \delta(t, t + s - 1) H_t^{ij}(t + s - 1; \rho_t, h_t) \right]^2} \cdot \left[ \sum_{s=1}^{T} \delta(t, t + s - 1) H_t^{ij}(t + s - 1; \rho_t, h_t) \right] \]

\[ \frac{d}{d\pi} \left[ \sum_{s=1}^{T} \delta(t, t + s - 1) H_t^{ij}(t + s - 1; \rho_t, h_t) \right] \frac{2}{\left[ \sum_{s=1}^{T} \delta(t, t + s - 1) H_t^{ij}(t + s - 1; \rho_t, h_t) \right]^2}|_0. \]

\[ \cdot \left[ \sum_{s=1}^{T} \delta(t, t + s) H_t^{ij}(t + s - 1; \rho_t, h_t)(h_{t+s}^i - (1 - S)h_{t+s}^{ij}) \right] \]

The second term, at the approximation point, is always 0. We can reduce the derivative to:

\[ \frac{d}{d\pi} G|_0 = \frac{\sum_{s=1}^{T} \delta(t, t + s) \frac{d}{d\pi}(h_{t+s}^i - (1 - S)h_{t+s}^{ij})}{\sum_{s=1}^{T} \delta(t, t + s - 1)} \]

Now,

\[ h_{t+s}^i = (\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s)(\pi_i + \pi_{ij}) \]

\[ h_{t+s}^{ij} = (\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)(\pi_j + \pi_{ij}) \]

\[ h_{t+s}^{ij} = (\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)\pi_{ij} \]

\[ \frac{d}{d\pi} (h_{t+s}^i - (1 - S)h_{t+s}^{ij}) = \]

\[ = \frac{d}{d\pi} ((\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s)(\pi_i + \pi_{ij}) - (1 - S)(\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)\pi_{ij}) \]
So that:
\[
\frac{d}{d\pi} (h^i_t + (1-S)h^i_t) = (\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s)
\]
and
\[
\frac{d}{d\pi} (h^i_{t+s} + (1-S)h^i_{t+s}) = (\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s) - (1-S)(\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)
\]
\[
\frac{d}{d\pi} (h^i_{t+s} + (1-S)h^i_{t+s}) = 0
\]
Therefore we have:
\[
\frac{d}{d\pi} G|_0 = \frac{\sum_{s=1}^{T} \delta(t,t+s) \frac{d}{d\pi} (h^i_t + (1-S)h^i_t)}{\sum_{s=1}^{T} \delta(t,t+s-1)}
\]
\[
G \simeq \sum_{s=1}^{T} \delta(t,t+s)(\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s) - (1-S)(\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)h^i_t
\]
Calling
\[
G_i = \frac{\sum_{s=1}^{T} \delta(t,t+s)(\alpha_{i,t} - \rho_{i,t}^s \alpha_{i,t} + \rho_{i,t}^s)}{\sum_{s=1}^{T} \delta(t,t+s-1)}
\]
and
\[
G_{ij} = \frac{\sum_{s=1}^{T} \delta(t,t+s)(\alpha_{ij,t} - \rho_{ij,t}^s \alpha_{ij,t} + \rho_{ij,t}^s)}{\sum_{s=1}^{T} \delta(t,t+s-1)}
\]
we obtain:
\[
\frac{\sigma_{ij}}{(1-R)} \simeq G_i h^i_t - (1-S)G_{ij} h^i_{t+s}
\]
Summing over counterparties, we have:
\[
\frac{(N-1)\sigma_i}{G_i(1-R)} \simeq (N-1)h^i_t - (1-S)\sum_{j \neq i} \frac{G_{ij}}{G_i} h^i_{t+s}
\]
which is again linear in the marginal and pairwise default probabilities and can be used in the LP formulation. To impose that CDS spreads inherit the shape of the hazard rates estimated from bonds, we can assume \(\alpha_{ij,t} = \alpha_{i,t}\) and \(\rho_{ij,t} = \rho_{i,t}\). Note that this reduces the CDS spread to a function of only one parameter: \(h^i_t\).

D.6 - Using interest rate swaps as the risk-free rate

While swap rates may not be the appropriate rate to discount cash flows under risk neutral probabilities (because they are indexed to a risky reference, LIBOR, and because they contain counterparty risk), it is interesting to check how the results change if we use them in place of Treasury rates.\(^{10}\) Because these rates are higher than the Treasury rates,\(^{10}\) I bootstrap the zero-coupon yield curve from the par swap rate curve of the different
and therefore result in a lower basis for all banks, we would expect the upper bounds on systemic risk to decrease noticeably. At the same time, remember that we are calibrating the time variation in the liquidity process to the basis of non-financial firms, and the level of the liquidity process to the basis of each bank in 2004. Therefore, the change in the risk-free rate will be offset by a corresponding decrease in the liquidity process (even though the offset is not exactly one to one). Appendix Table 1 shows that the change in the bounds is very small.

D.7 - Assumptions about the weighting of contributors in CDS contracts

As discussed in section 4, the bounds are computed under the assumption that the CDS spreads are obtained by averaging quotes obtained from all the other dealers in the sample. If some dealers do not post quotes at all times, the average spread observed will, in expectation, overweight dealers which send quotes in more frequently. In turn, this is most likely related to how active the dealer is in the CDS market.

While we cannot obtain directly estimates of the activity of the dealers (in terms of number of contracts written and volume of CDS protection sold), Fitch Ratings\textsuperscript{11} reports a ranking of the top 5 counterparties by trade count (which in turn is very correlated with gross positions sold), for each year between 2006 and 2010. We might then think that because these dealers are more active, quotes are more likely to be obtained from them, and therefore the average CDS spread observed will in expectation reflect more their contribution. Given this, as a robustness test I compute bounds that overweight the top 5 institutions in the formula for CDS contracts. I consider two relatively extreme weighting schemes. In all of them, institutions ranked below 5 have the same weight (I do not have information about the relative ranking of these dealers). In the first weighting scheme, I compute the bounds assuming that the top 5 institutions are 5 times more likely than the other 10 to contribute quotes, and therefore their contribution is weighted 5 times more than the other institutions in the sample. The second weighting scheme again assumes that all institutions ranked 6-15 have the same weight, and the top dealer has 10 times their weight, the second dealer 8 times, and so on up to the 5th largest dealer (with a weight twice that of the smaller dealers).

The effect of this overweighting on the bounds of systemic risk is not immediate. Suppose, for example, that in the bounds computed under equal weighting, systemic risk comes from the joint default risk among top-5 banks. Then, increasing the weight on these banks will have the effect, everything else constant, of lowering the weighted CDS spreads. But this is not possible because the CDS spreads were chosen to match the observed ones. Therefore, the joint default risk among these banks will have to decrease. At the same time, joint default risk with smaller banks can increase. But if these smaller banks were contributing little to default risk before the change in weights, an increase in the possibility of joint default risk with them might not make up for the reduction in maximum systemic risk coming from the top-5 dealers. In this example, systemic risk will likely decrease when we overweight top-5 dealers. It is easy to see that the opposite is true if systemic risk mainly comes from non top-5 dealers.

\textsuperscript{11}Fitch Ratings, 2008, Global Credit Derivatives Survey.
If instead in both groups (top-5 and non-top-5 dealers) we find dealers with large contribution to systemic risk as well as dealers with small contribution to systemic risk, under equal weighting, the bounds will be relatively robust to changes in the weights. In fact, this is the case. The top 5 banks include both banks with high contribution to systemic risk as well as banks with low contribution to systemic risk, such as one or two European banks. Appendix Table 1 shows that under both weighting schemes the main results still hold.

This robustness test also allows us to say something about heterogeneity in collateral agreements across counterparties. All the results in the paper have been derived assuming that the recovery rate in case of double default, $S$, is the same for all banks. How do the main results change if instead (because of different collateral agreements and exposure to other shocks) the recovery rate is different across institutions? While we have no direct information about the expected recovery rates of each counterparty, it is easy to show that if the recovery rates $S_j$ are different across counterparties $j$, the average quote reflects not an equally weighted average across $j$’s of the joint default probabilities $P(A_i \cap A_j)$, but rather a weighted average $\sum w_j P(A_i \cap A_j)$, where $w_j = \frac{(1-S_j)}{(N-1)(1-S)}$ and $\bar{S} = \frac{1}{N-1} \sum_j S_j$. Therefore, given a certain average recovery rate $\bar{S}$, the joint default risk with counterparty $j$ will be weighted more in the observed quote if $j$’s recovery rate is lower. Now, it is reasonable to assume that more important counterparties (that have a larger volume of the business) are also the counterparties that are able to obtain less stringent collateral agreements – and therefore buyers of CDSs from them might obtain a lower recovery rate in case of double default. As a consequence, the robustness test presented in this section can also be interpreted as robustness to this case of heterogeneity in recovery rates.

D.8 - Assumptions about the exchange rate

The construction of the bounds on systemic risk involves the estimation of risk-neutral probabilities from bond prices and of joint default probabilities from CDS spreads. Using probabilities obtained from different securities to obtain risk-neutral probabilities of joint default requires additional assumptions if the securities are denominated in different currencies. In particular, while most bonds issued by American firms and the CDSs written on them are denominated in dollars, European firms issue several bonds in Euros and in other currencies, and the CDSs written on them are denominated in Euros.

To simplify the discussion, consider one-period bonds and CDSs written by banks $i$ and $j$. Call $m_{se}$ the stochastic discount factor of a US investor in state $(s,e)$. Here, $s$ indicates the default state of the banks $i$ and $j$, so that it can take values $i$ (only $i$ defaults), $j$ (only $j$ defaults), $ij$ (both default), and 0 (none defaults). $e$ indicates the exchange rate with a foreign currency. Call $\pi_s$ the probability of $s$ occurring, and note that $\pi_s E[m_{se}|s]$ is the price of a security that pays 1 if default state $s$ happens. The price of a state-contingent security that pays a unit of foreign currency if default state $s$ happens is then $\pi_s E[e m_{se}|s]$.

It is easy to see that a sufficient condition for correctly estimating risk-neutral default probabilities using bonds and CDSs denominated in different currencies (using the risk-free rates denominated in the respective currencies to discount cash flows) is that for each $s$:

$$\frac{E[e \cdot m_{se}|s]}{E[m_{se}|s]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]}$$ (3)
which requires that the relative price of domestic and foreign risk-free securities is the same as the relative price of domestic and foreign state-contingent securities that pay off in the various default states. Of course, it is reasonable to assume that the relative price of dollar-denominated and foreign currency-denominated default-contingent securities might be different depending on the default state (think for example of a flight-to-quality to US securities if several banks default). As a robustness test for the validity of the bounds in case these conditions are violated, I perform the estimation exercise including only American firms, for which all bonds and CDSs are dollar-denominated. Appendix Table 1 shows that the results still hold for this subset of banks.

To derive equation (3), we can start by considering the price of a bond issued by $i$ denominated in different currencies. Consider the variable $d_i$ which indicates $i$’s default, and is 1 if $s = i$ or $s = ij$, and 0 otherwise. Call $\pi_{d_i} = 1$ the probability that $i$ defaults.

Note that $\pi_{d_i} = 1 = E[m_{se}|d_i = 1] + \pi_{d_i} = 0 E[m_{se}|d_i = 0] = E[m_{se}]$.

For a dollar-denominated risky bond (R is the recovery rate), the dollar price is:

$$p_i^d = \pi_{d_i = 0} E[m_{se}|d_i = 0] + R \pi_{d_i = 1} E[m_{se}|d_i = 1] = E[m_{se}] - (1 - R) \pi_{d_i = 1} E[m_{se}|d_i = 1]$$

Now consider a euro-denominated bond issued by the same firm, and of equal seniority. Calling $e_0$ the time-0 exchange rate, we obtain:

$$p_i^e e_0 = \pi_{d_i = 0} E[e \cdot m_{se}|d_i = 0] + R \pi_{d_i = 1} E[e \cdot m_{se}|d_i = 1] = E[e \cdot m_{se}] - (1 - R) \pi_{d_i = 1} E[e \cdot m_{se}|d_i = 1]$$

The prices of the respective risk-free securities are:

$$t^d = E[m_{se}]$$

$$t^e e_0 = E[e \cdot m_{se}]$$

Combining defaultable and risk-free bonds we get:

$$p_i^d = t^d \left(1 - (1 - R) \pi_{d_i = 1} \frac{E[m_{se}|d_i = 1]}{E[m]} \right)$$

$$p_i^e e_0 = t^e e_0 \left(1 - (1 - R) \pi_{d_i = 1} \frac{E[e \cdot m_{se}|d_i = 1]}{E[e \cdot m_{se}]} \right)$$

We can then use either bond to estimate the risk-neutral probability of default of firm $i$

$$P(A_i) = \pi_{d_i = 1} \frac{E[m_{se}|d_i = 1]}{E[m_{se}]}$$

discounting cash flows by the appropriate risk-free rate as long as the following condition holds:

$$\frac{E[e \cdot m_{se}|d_i]}{E[m_{se}|d_i]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]}$$  (4)
which requires that the relative price of domestic and foreign risk-free securities is the same as
the relative price of domestic and foreign state-contingent securities that pay off if \( i \) defaults.

Now, consider the case of a CDS written by one bank on the default of another bank. The
CDS is written on a European bank \( (i) \) but the counterparty \( (j) \) is American. The contract
is denominated in euros.

The CDS contract costs \( z_{ji} \) euros. So we must have

\[
z_{ji}e_0 = E[e \cdot m_{se}] \left((1 - R)\pi_i \frac{E[e \cdot m_{se} | s = i]}{E[e \cdot m_{se}]} + (1 - R)S_{ij} \frac{E[e \cdot m_{se} | s = ij]}{E[e \cdot m_{se}]}\right)
\]

Therefore, as long as the European yield curve is used to discount cash flows for euro-
denominated CDSs the sufficient condition is:

\[
\frac{E[e \cdot m_{se} | s]}{E[m_{se} | s]} = \frac{E[e \cdot m_{se}]}{E[m_{se}]}
\]

for every default event in \( s \). Note that equation (5) implies (4) so it is a sufficient condition
for both bonds and CDSs of different currencies.

D.9 - Using only larger transactions from TRACE

A concern with using bond prices from Bloomberg is that they might incorporate stale
information (for European bonds, for which I use quoted prices), or they might depend on
very small trades, which might be less reflective of credit risk (see for example Dick-Nielsen
et al. (2010)). To make sure results are robust to these problems, I compute the bounds
for the subset of US firms using only transaction data from TRACE, and ignoring all trades
with nominal amounts of less than $100,000. Of course, this will exclude several bonds for
several days. Appendix Table 1 reports that the bounds change very little.
### Appendix Tables

#### Appendix Table 1a: max P1

<table>
<thead>
<tr>
<th>Model</th>
<th>2007</th>
<th>Jan 2008 to Bear</th>
<th>Bear to Lehman</th>
<th>Month after Lehman</th>
<th>Oct 2008 to April 2009</th>
<th>After April 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>S</td>
<td>Date</td>
<td>Date</td>
<td>Date</td>
<td>Date</td>
</tr>
<tr>
<td></td>
<td>1/1/07</td>
<td>1/1/08</td>
<td>3/16/08</td>
<td>9/15/08</td>
<td>10/16/08</td>
<td>5/1/09</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.10</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.30</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.40</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.70</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.90</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.10</td>
<td>1.00</td>
<td>50.4</td>
<td>178.0</td>
<td>168.8</td>
<td>298.1</td>
<td>221.7</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.30</td>
<td>64.8</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.0</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.40</td>
<td>64.8</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.0</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.70</td>
<td>64.8</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.0</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.90</td>
<td>64.8</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.0</td>
</tr>
<tr>
<td>R 0.30</td>
<td>1.00</td>
<td>64.8</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.0</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.40</td>
<td>75.6</td>
<td>267.1</td>
<td>253.2</td>
<td>447.1</td>
<td>332.5</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.70</td>
<td>75.6</td>
<td>267.1</td>
<td>253.2</td>
<td>447.1</td>
<td>332.5</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.90</td>
<td>75.6</td>
<td>267.1</td>
<td>253.2</td>
<td>447.1</td>
<td>332.5</td>
</tr>
<tr>
<td>R 0.40</td>
<td>1.00</td>
<td>75.6</td>
<td>267.1</td>
<td>253.2</td>
<td>447.1</td>
<td>332.5</td>
</tr>
<tr>
<td>Using swap rates</td>
<td></td>
<td>64.8</td>
<td>229.0</td>
<td>217.1</td>
<td>383.6</td>
<td>285.2</td>
</tr>
<tr>
<td>US banks</td>
<td></td>
<td>49.4</td>
<td>166.7</td>
<td>156.7</td>
<td>278.7</td>
<td>182.7</td>
</tr>
<tr>
<td>US banks, larger trans</td>
<td></td>
<td>46.5</td>
<td>166.4</td>
<td>156.6</td>
<td>278.7</td>
<td>183.3</td>
</tr>
<tr>
<td>Reweight top 5 banks</td>
<td></td>
<td>65.0</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.8</td>
</tr>
<tr>
<td>Reweight, decreasing</td>
<td></td>
<td>65.0</td>
<td>228.9</td>
<td>217.0</td>
<td>383.3</td>
<td>285.8</td>
</tr>
<tr>
<td>Alternative bond model</td>
<td></td>
<td>35.2</td>
<td>112.0</td>
<td>162.2</td>
<td>579.0</td>
<td>216.9</td>
</tr>
</tbody>
</table>

#### Appendix Table 1b: max P4

<table>
<thead>
<tr>
<th>Model</th>
<th>2007</th>
<th>Jan 2008 to Bear</th>
<th>Bear to Lehman</th>
<th>Month after Lehman</th>
<th>Oct 2008 to April 2009</th>
<th>After April 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>S</td>
<td>Date</td>
<td>Date</td>
<td>Date</td>
<td>Date</td>
</tr>
<tr>
<td></td>
<td>1/1/07</td>
<td>1/1/08</td>
<td>3/16/08</td>
<td>9/15/08</td>
<td>10/16/08</td>
<td>5/1/09</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.10</td>
<td>2.2</td>
<td>2.3</td>
<td>15.7</td>
<td>24.3</td>
<td>49.1</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.30</td>
<td>2.4</td>
<td>2.5</td>
<td>17.3</td>
<td>25.7</td>
<td>48.8</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.40</td>
<td>2.5</td>
<td>2.6</td>
<td>18.2</td>
<td>26.2</td>
<td>48.6</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.70</td>
<td>2.8</td>
<td>2.9</td>
<td>20.0</td>
<td>27.7</td>
<td>47.2</td>
</tr>
<tr>
<td>R 0.10</td>
<td>0.90</td>
<td>3.3</td>
<td>3.3</td>
<td>21.5</td>
<td>28.4</td>
<td>46.0</td>
</tr>
<tr>
<td>R 0.10</td>
<td>1.00</td>
<td>12.6</td>
<td>44.5</td>
<td>42.2</td>
<td>70.0</td>
<td>55.4</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.30</td>
<td>3.3</td>
<td>3.5</td>
<td>24.7</td>
<td>42.1</td>
<td>64.8</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.40</td>
<td>3.4</td>
<td>3.7</td>
<td>25.6</td>
<td>42.5</td>
<td>64.2</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.70</td>
<td>3.7</td>
<td>4.1</td>
<td>27.9</td>
<td>43.7</td>
<td>62.0</td>
</tr>
<tr>
<td>R 0.30</td>
<td>0.90</td>
<td>4.1</td>
<td>4.6</td>
<td>29.5</td>
<td>44.3</td>
<td>60.6</td>
</tr>
<tr>
<td>R 0.30</td>
<td>1.00</td>
<td>16.2</td>
<td>57.2</td>
<td>54.3</td>
<td>90.0</td>
<td>71.2</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.40</td>
<td>4.0</td>
<td>4.4</td>
<td>32.0</td>
<td>54.0</td>
<td>76.8</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.70</td>
<td>4.3</td>
<td>5.0</td>
<td>34.7</td>
<td>54.0</td>
<td>74.0</td>
</tr>
<tr>
<td>R 0.40</td>
<td>0.90</td>
<td>4.8</td>
<td>5.7</td>
<td>36.1</td>
<td>53.3</td>
<td>72.0</td>
</tr>
<tr>
<td>R 0.40</td>
<td>1.00</td>
<td>18.9</td>
<td>66.8</td>
<td>63.3</td>
<td>104.9</td>
<td>83.1</td>
</tr>
<tr>
<td>Using swap rates</td>
<td></td>
<td>2.3</td>
<td>2.6</td>
<td>19.3</td>
<td>42.5</td>
<td>58.1</td>
</tr>
<tr>
<td>US banks</td>
<td></td>
<td>1.3</td>
<td>0.8</td>
<td>11.3</td>
<td>36.9</td>
<td>36.0</td>
</tr>
<tr>
<td>US banks, larger trans</td>
<td></td>
<td>1.4</td>
<td>1.0</td>
<td>16.6</td>
<td>39.6</td>
<td>42.1</td>
</tr>
<tr>
<td>Reweight top 5 banks</td>
<td></td>
<td>5.3</td>
<td>5.0</td>
<td>32.4</td>
<td>50.3</td>
<td>74.5</td>
</tr>
<tr>
<td>Reweight, decreasing</td>
<td></td>
<td>5.2</td>
<td>5.1</td>
<td>32.9</td>
<td>50.1</td>
<td>75.6</td>
</tr>
<tr>
<td>Alternative bond model</td>
<td></td>
<td>2.6</td>
<td>4.7</td>
<td>18.7</td>
<td>49.5</td>
<td>36.7</td>
</tr>
</tbody>
</table>
### Appendix Table 1c: min P1

<table>
<thead>
<tr>
<th>Model</th>
<th>Average level of the bounds (bp per month)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2007</td>
</tr>
<tr>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>0.10</td>
<td>0.30</td>
</tr>
<tr>
<td>0.10</td>
<td>0.40</td>
</tr>
<tr>
<td>0.10</td>
<td>0.70</td>
</tr>
<tr>
<td>0.10</td>
<td>0.90</td>
</tr>
<tr>
<td>0.10</td>
<td>1.00</td>
</tr>
<tr>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>0.30</td>
<td>0.40</td>
</tr>
<tr>
<td>0.30</td>
<td>0.70</td>
</tr>
<tr>
<td>0.30</td>
<td>0.90</td>
</tr>
<tr>
<td>0.30</td>
<td>1.00</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.40</td>
<td>0.70</td>
</tr>
<tr>
<td>0.40</td>
<td>0.90</td>
</tr>
<tr>
<td>0.40</td>
<td>1.00</td>
</tr>
<tr>
<td>Using swap rates</td>
<td>56.6</td>
</tr>
<tr>
<td>US banks</td>
<td>43.3</td>
</tr>
<tr>
<td>US banks, larger trans</td>
<td>39.8</td>
</tr>
<tr>
<td>Reweight top 5 banks</td>
<td>50.4</td>
</tr>
<tr>
<td>Reweight, decreasing</td>
<td>50.9</td>
</tr>
<tr>
<td>Alternative bond model</td>
<td>25.2</td>
</tr>
</tbody>
</table>

Note: Table reports the average value of the bounds on monthly P1 (probability that at least one bank defaults) and P4 (probability that at least four banks default) for different nonoverlapping periods, under different assumptions discussed in the text. The lower bound for P4 is 0 throughout.