Outline

- Conic optimization
- Second order cone optimization example: robust linear programming;
- Semidefinite programming examples: Lyapunov stability and data fitting;
- Software.
The set $K \subset \mathbb{R}^n$ is a \textit{convex cone} if it is a convex set and for all $x \in K$ and $\lambda > 0$ one has $\lambda x \in K$. 

\textbf{Cones}
Conic optimization problem

Data:
- A convex cone $K \subset \mathbb{R}^n$;
- A linear operator $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$;
- Vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, and an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$.

Conic optimization problem

$$\inf_{x \in K} \{ \langle c, x \rangle : Ax = b \}.$$
Choices for $K$

We consider the conic optimization problem for three choices of the cone $K$ (or Cartesian products of cones of this type):

- **Linear Programming (LP):** $K$ is the nonnegative orthant in $\mathbb{R}^n$:
  \[
  \mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0 \ (i = 1, \ldots, n)\},
  \]

- **Second order cone programming (SOCP):** $K$ is the second order (Lorentz) cone:
  \[
  \left\{ \begin{bmatrix} x \\ t \end{bmatrix} : x \in \mathbb{R}^n, \ t \in \mathbb{R}, \ t \geq \|x\| \right\}.
  \]

- **Semidefinite programming (SDP):** $K$ is the cone of symmetric positive semidefinite matrices.
We consider an LP problem with ‘uncertain’ data.

Robust LP Problem

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \\
& a_i^T x \leq b_i \ (i = 1, \ldots, m) \quad \forall a_i \in \mathcal{E}_i \ (i = 1, \ldots, m),
\end{align*}
\]

where the \( \mathcal{E}_i \) are given ellipsoids:

\[
\mathcal{E}_i = \{ \bar{a}_i + P_i u : \|u\| \leq 1 \},
\]

with \( P_i \) symmetric positive semidefinite.
Robust LP: SOCP formulation

We had

$$\mathcal{E}_i := \{ \bar{a}_i + P_i u : \|u\| \leq 1 \}.$$ 

Notice that

$$a_i^T x \leq b_i \ \forall a_i \in \mathcal{E}_i \iff \bar{a}_i^T x + \|P_i x\| \leq b_i$$

Robust LP Problem: SOCP reformulation

$$\min c^T x$$

subject to

$$\bar{a}_i^T x + \|P_i x\| \leq b_i \ (i = 1, \ldots, m).$$

Note that this is indeed an SOCP problem.
The solutions of at least 13 of the 90 Netlib LP problems are meaningless if there is \textbf{0.01\% uncertainty} in the data entries! Solving the robust LP instead overcomes this difficulty.


More SOCP examples in the online paper:


Applications include robust least squares problems, portfolio selection, filter design ...
Example: sum of squares polynomials

Example

Is \( p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \) a sum of squared polynomials?

**YES**, because

\[
p(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.
\]

The 3 \( \times \) 3 matrix (say \( M \)) is positive semidefinite and:

\[
M = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},
\]

and consequently

\[
p(x) = \frac{1}{2} \left( 2x_1^2 - 3x_2^2 + x_1x_2 \right)^2 + \frac{1}{2} \left( x_2^2 + 3x_1x_2 \right)^2.
\]
The example illustrates the fact that deciding if a polynomial is a sum of squares is equivalent to an SDP problem;

This has application in polynomial optimization problems, ...

... data fitting using nonnegative or monotone polynomials,

... and finding polynomial Lyapunov functions to prove stability of dynamical systems.
Example: Lyapunov stability

**Definition**

The origin is **asymptotically stable** for a dynamical system

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \]

if \( \lim_{t \to \infty} x(t) = 0 \) whenever \( x_0 \) is sufficiently close to 0. A sufficient condition for stability is a nonnegative Lyapunov function \( V : \mathbb{R}^n \to \mathbb{R} \) such that \( V(0) = 0 \) and \( \nabla V(x)^T f(x) < 0 \) if \( x \neq 0 \).

**Example (Parrilo):**

\[ \dot{x}_1(t) = -x_2(t) + \frac{3}{2} x_1^2(t) - \frac{1}{2} x_1^3(t) \]
\[ \dot{x}_2(t) = 3x_1(t) - x_2(t). \]

- Using SDP, one may find a degree 4 polynomial \( V \) to prove stability, ...
- ... where both \( V(x) \) and \( -\nabla V(x)^T f(x) \) are sums of squares.
Example: Lyapunov stability (ctd.)

- contours of $V(x)$;
- trajectories.
Example: Nonnegative data fitting
Lyapunov stability example from:


Data fitting example from:


Other SDP applications include free material optimization, sensor network localization, low rank matrix completion, ...
Free material optimization: wing design of the Airbus A380

Further reading

Software

Software that implements interior point methods for conic programming:

- Commercial LP solvers: CPLEX, MOSEK, XPRESS-MP, ...
- SOCP solvers: MOSEK, LOQO, SeDuMi
- SDP solvers: SDPT3, SeDuMi, CSDP, SDPA ...

Sizes of problems that can be solved in “reasonable time” (sparse data in the LP/SOCP case):

<table>
<thead>
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<th>LP</th>
<th>SOCP</th>
<th>SDP</th>
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<tr>
<td>n</td>
<td>$10^6$–$10^8$</td>
<td>$10^5$–$10^6$</td>
<td>$10^3 \times 10^3$</td>
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<td>m</td>
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